MAPPING SPACES OF Gray-CATEGORIES

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Abstract. We define a mapping space for Gray-enriched categories adapted to higher gauge theory. Our construction differs significantly from the canonical mapping space of enriched categories in that it is much less rigid. The two essential ingredients are a path space construction for Gray-categories and a kind of comonadic resolution of the 1-dimensional structure of a given Gray-category obtained by lifting the resolution of ordinary categories along the canonical fibration of GrayCat over Cat.

Contents

1. Introduction

Folk knowledge of yore, among algebraic models for homotopy n-types Gray-groupoids model 3-types; Lack [2011] gives us a proof using a model category methods. Wanting to study the homotopy 3-type of the moduli space of 3-connections on a manifold, we thought it apt to define a mapping space $[\mathcal{S}_3(M), \mathcal{C}(\mathcal{H})]$ of Gray-groupoids that could model that moduli space, where $\mathcal{S}_3(M)$ is the fundamental Gray-groupoid and $\mathcal{C}(\mathcal{H})$ is the Gray-groupoid ultimately derived from a 2-crossed Lie-algebra where the triconnections take their values; this is the obvious next step after 2-connections, see for example Schreiber

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and Waldorf. See [Martins and Picken 2011] for the background on the fundamental Gray-groupoid and triconnections.

In 1999 Crans gave a partial solution the mapping space problem; however, the absence of an interchange law in Gray-categories prevents lax transformations between Gray-functors from being composable in general. The slightly unsatisfactory solution is to restrict to those transformations and higher cells that can in fact be composed; this does give mapping space Gray-category, but a mere stopgap not sufficient for our purposes.

Instead we enlarge the repertoire on maps, and thereby transformations, in a way that will permit forming all composites of transformations; specifically we introduce a 2-cocycle that intermediates coherently between the two possible evaluations of arrangements of squares shown in (116) and (117). In analogy with Garner [2010] we introduce a comonadic weakening of strict Gray-functors in section 3. The comonad Q^1 then yields a co-Kleisli category $\mathsf{GrayCat}_{\mathrm{Q}^1}$. We use in an essential way that $\mathsf{GrayCat}$ is fibered over Cat.

Inspired by [Bénabou et al. 1967] we axiomatise lax transformations by maps into a path-space. In section 4 we introduce a functorial path-space construction for Gray-categories; subsequently in section 5 it is shown that this yields an internal category $\overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}$ in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ for a given $\mathbb{H}$ in $\mathsf{GrayCat}$.

The n-th iterate of $\overrightarrow{(\_)}$ yields an n-truncated internal cubical object in $\mathsf{GrayCat}$. In section 6 we construct an internal Gray-category

$$\overline{\overline{\overline{\mathbb{H}}}} \rightrightarrows \overline{\overline{\mathbb{H}}} \rightrightarrows \overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}$$

in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as a subobject of the third iterated path-space. It is then a trivial consequence that we obtain a mapping space Gray-category by applying the hom functor

$$[\mathbb{G}, \mathbb{H}] := \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overline{\overline{\overline{\mathbb{H}}}} \rightrightarrows \overline{\overline{\mathbb{H}}} \rightrightarrows \overrightarrow{\mathbb{H}} \rightrightarrows \mathbb{H}).$$

We hope to be able to prove in a later paper that this internal hom is part of a monoidal closed structure on $\mathsf{GrayCat}_{\mathrm{Q}^1}$ involving a suitable extension of Crans' tensor product.

Lastly, we remark that if $\mathbb{H}$ is a Gray-groupoid then $\overrightarrow{\mathbb{H}}$ as well as $[\mathbb{G}, \mathbb{H}]$ will be Gray-groupoids.

2. Gray-Categories

Our principal objects of study in this paper are the following. For explicit details see [Crans 1999].

2.1. Definition. *A* **Gray-category** *is a category enriched in the category* Gray *of 2-categories with the Gray-tensor product.*

Even though this a nice and concise definition for our purposes it is not flexible enough, we need a more elementary way of treating Gray-categories. We thus provide a definition in the internal language of a category with finite limits in appendix A. For now we just make the following observation:

2.2. Remark. [[Crans 1999, 2.3]] *A* Gray*-category is a reflexive 3-globular set* $\mathbb{G}_{0,\ldots,3}$*, with composition operations* $\#_k$*, where* k *denotes the dimension of the incidence cell.*

In general we can say that composing an i*-cell with a* j*-cell along a* k*-cell yields a* $i+j-(k+1)$*-cell. The ones where* $i=j$ *and* $k=i-1$ *are called vertical. The ones where* $i+j-(k+1)=\max\{i,j\}$ *are called whiskers.*

2.3. Definition. [[Crans 1999, 5.1]] *A* Gray**-functor** *between* Gray*-categories is a map of globular sets, that preserves all the above operations.*

3. Resolution in Dimension One

We define a resolution of the 1-dimensional structure of a Gray-category using a comonad, by lifting the free category comonad called "path" in [Dawson et al. 2006] to Gray-categories; but note that we use the term in a different way in this paper.

The resulting co-Kleisli category can be seen as the category of Gray-categories with an enlarged repertoire of maps, that is flexible enough to carry out our path space construction. After giving an abstract construction of this category of pseudo maps we proceed to characterize them explicitly.

3.1. Basic Fibrations. There are obvious functors

$$\mathsf{GrayCat} \xrightarrow{(\_)_2} \mathsf{SesquiCat} \xrightarrow{(\_)_1} \mathsf{Cat} \xrightarrow{(\_)_0} \mathsf{Set} \tag{1}$$

that forget the 3-cells, the 2-cells and 1-cells respectively. The last one will not play an explicit role here.

Let $\mathfrak{S}$ be a sesquicategory, $\mathbb{G}$ a Gray-category, and $F\colon \mathfrak{S} \longrightarrow \mathbb{G}_2$ a sesquifunctor. We define $\overline{F}\colon F^*\mathfrak{S} \longrightarrow \mathbb{G}$ as follows:

$$(F^*\mathfrak{S})_0 = \mathfrak{S}_0 \tag{2}$$
$$(F^*\mathfrak{S})_1 = \mathfrak{S}_1 \tag{3}$$
$$(F^*\mathfrak{S})_2 = \mathfrak{S}_2 \tag{4}$$
$$(F^*\mathfrak{S})_3 = \{(\Gamma;\alpha,\beta)\,|\,\Gamma\colon F\alpha \longrightarrow F\beta\} \tag{5}$$

Note that the interchange of two 2-cells α,β in $F^*\mathfrak{S}$ incident on a 0-cell is given essentially by the interchange of their images under F:

$$\beta{\otimes}\alpha = (F\beta{\otimes}F\beta;\beta \triangleright \alpha, \beta \triangleleft \alpha)\,. \tag{6}$$

Let us take note of the following useful fact that helps to characterize the Cartesian maps:

3.2. Remark. *For a functor $p\colon \mathsf{E} \longrightarrow \mathsf{B}$ that preserves co-limits, let $D\colon \mathsf{D} \longrightarrow \mathsf{E}$ a diagram in E with co-limit (C, k_i)*

$$\begin{tikzcd} D_i \arrow[r, "k_i"] & C \arrow[drr, "g"] & & \\ & & A \arrow[r, "f"'] & B \end{tikzcd}, \tag{7}$$

assume $p(g)$ factors below as $p(f)u = p(g)$. Furthermore, assume that the induced sink $(u_i) = up(k_i)$ has fillers $\langle u_i \rangle$ above with $f\,\langle u_i \rangle = gk_i$, then the co-universally induced map $\langle u \rangle : C \longrightarrow A$ is a filler over u.

This means that to check whether a map f is Cartesian we don't need to give the filler u directly, but we can define it on presumably simpler parts of C. These then combine into a valid filler.

3.3. Remark. *Maps Cartesian with respect to $(\_)_2$ are exactly the Gray-functors, that are 2-locally isomorphisms of sets. That is, given two parallel 2-cells on the intervening 3-cells the map is bijective.*

3.4. Lemma. *$F^*\mathfrak{S}$ is a Gray-category, $\overline{F}$ is a Gray-functor and Cartesian with respect to $(\_)_2$.* □

Similarly, let $\mathfrak{S}$ a sesquicategory and C a category, $F\colon \mathsf{C} \longrightarrow \mathfrak{S}_1$ a functor, then we define a sesquicategory:

$$(F^*\mathsf{C})_0 = \mathsf{C}_0 \tag{8}$$
$$(F^*\mathsf{C})_1 = \mathsf{C}_1 \tag{9}$$
$$(F^*\mathsf{C})_2 = \{(\alpha; f, g) | \alpha\colon Ff \longrightarrow Fg\} \tag{10}$$

3.5. Lemma. *$F^*\mathsf{C}$ is a sesquicategory, $\overline{F}$ is a sesquifunctor, and Cartesian with respect to $(\_)_1$.* □

3.6. Remark. *Maps Cartesian with respect to $(\_)_1$ are exactly the sesquifunctors, that are 1-locally isomorphisms of sets. That is, given two parallel 1-cells on the intervening 2-cells the map is bijective.*

We will denote the composite $(\_)_1(\_)_2$ also by $(\_)_1$, it is of course a fibration as well. For later reference we describe its Cartesian liftings explicitly as well. Let $\mathbb{G}$ be a Gray-category, $\mathbb{G}_1$ its underlying category. Let C be an ordinary category and $F\colon \mathsf{C} \longrightarrow \mathbb{G}_1$ a functor. Then $F^*\mathbb{G}$ is given by:

$$(F^*\mathbb{G})_0 = \mathsf{C}_0 \tag{11}$$
$$(F^*\mathbb{G})_1 = \mathsf{C}_1 \tag{12}$$
$$(F^*\mathbb{G})_2 = \{(\alpha; f, g) | f, g\colon x \longrightarrow y,\ \alpha\colon Ff \longrightarrow Fg\} \tag{13}$$
$$(F^*\mathbb{G})_3 = \{(\Gamma; \alpha, \beta; f, g) | f, g\colon x \longrightarrow y,\ \Gamma\colon F\alpha \longrightarrow F\beta\} \tag{14}$$

Source and target maps are as follows:

$$s_2(\Gamma;\alpha,\beta;f,g) = (\alpha;f,g) \qquad\qquad t_2(\Gamma;\alpha,\beta;f,g) = (\beta;f,g) \tag{15}$$

$$s_1(\alpha;f,g) = f \qquad\qquad t_1(\alpha;f,g) = g\,. \tag{16}$$

and s_0, t_0 are as given by C. As identities we take:

$$i_1(f) = (\mathrm{id}_{Ff};f,f) \quad i_2(\alpha;f,g) = (\mathrm{id}_\alpha;\alpha,\alpha,f,g)\,. \tag{17}$$

The tensor in $F^*\mathbb{G}$ of two 2-cells is

$$(\beta;g,g')\otimes(\alpha;f,f') = (\beta\otimes\alpha;\beta\triangleleft\alpha,\beta\triangleright\alpha;g\#_0 f,g'\#_0 f') \tag{18}$$

where

$$\beta\triangleleft\alpha = (\beta\#_0 Ff')\#_1(Fg\#_0\alpha), \quad \beta\triangleright\alpha = (Fg'\#_0\alpha)\#_1(\beta\#_1 Ff)\,. \tag{19}$$

There is an obvious map $\overline{F}\colon F^*\mathbb{G} \longrightarrow \mathbb{G}$ over F that acts like F on 0- and 1-cells, and on 2- and 3-cells as a projection to $\mathbb{G}$.

3.7. Remark. *The globular set $F^*\mathbb{G}$ is a Gray-category. The composition operations of $F^*\mathbb{G}$ are given by those of C and $\mathbb{G}$ and it is easy to see that they fulfill the axioms of a Gray-category.*

Obviously $G^*F^*\mathbb{G} \cong (FG)^*\mathbb{G}$ and $\mathrm{id}_{\mathsf{C}}^* \cong \mathrm{id}_{\mathsf{GrayCat}_{\mathsf{C}}}$ coherently. Also, we can always choose $\mathrm{id}_{\mathsf{C}}^* = \mathrm{id}_{\mathsf{GrayCat}_{\mathsf{C}}}$, but this is not necessary in what follows.

3.8. Lemma. *A map of Gray-categories is Cartesian with respect to $\mathbb{G} \mapsto \mathbb{G}_1$ iff it is 1-locally an isomorphism of categories, i.e. given two parallel 1-cells the map is bijective on the intervening 2-cells and in turn bijective on the 3-cells between parallel such.* □

3.9. Definition. *We define a map of Gray-categories to be an n-**isomorphism** if it is Cartesian with respect to $(\_)_n$. It is n-**faithful** if fillers of factorizations under $(\_)_n$ are unique, and n-**full** is there (not necessarily unique) fillers for all factorizations under $(\_)_n$.*

With this definition 0-fidelity is ordinary fidelity of functors, 1-fidelity is local fidelity, and so on.

3.10. Remark. *One property of Cartesian maps in a fibration p that we are going to exploit in the proof of the following theorem is that for three arrows upstairs,*

$$\begin{tikzcd} \cdot \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & \cdot \arrow[r, "f"] & \cdot \end{tikzcd} \tag{20}$$

with f Cartesian, $p(r) = p(s)$ downstairs and $fr = fs$ upstairs imply $r = s$, on account of f being p-faithful.

3.11. Lemma. *If fg is Cartesian with respect to a given fibration p and f is p-faithful, then g is p-Cartesian.*

Proof Assume k and u such that $p(g)u = p(k)$, then $p(fg)u = p(fk)$ and hence by fg being p-full there is a filler $\langle u \rangle$ such that $fg\langle u \rangle = fk$. Then by f being p-faithful $g\langle u \rangle = k$.

By fg being p-faithful $\langle u \rangle$ is the unique such filler. □

3.12. Comonad Liftings.

3.13. Theorem. *Given a fibration of categories $p\colon \mathsf{E} \longrightarrow \mathsf{B}$, a comonad (Q, δ, ε) on B can be lifted to a comonad (K, d, e) on E such that $(K, Q)\colon p \longrightarrow p$ is a comonad in the 2-category of all fibrations.*

Proof Let $(\_)^*\colon \mathsf{B}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ be a chosen cleavage. For every $A \in \mathsf{E}_x$ we let $e_A\colon (KA = \varepsilon_x^* A) \longrightarrow A$ be the chosen Cartesian lift of $\varepsilon_x\colon Qx \longrightarrow x$. For a morphism f over j in

$$\begin{tikzcd} KA \arrow[r, "e_A"] \arrow[dr, dotted, "Kf"'] & A \arrow[dr, "f"] & \\ & KB \arrow[r, "e_B"'] & B \end{tikzcd} \tag{21}$$

$$\begin{tikzcd} Qx \arrow[r, "\varepsilon_x"] \arrow[dr, "Qj"'] & x \arrow[dr, "j"] & \\ & Qy \arrow[r, "\varepsilon_y"'] & y \end{tikzcd}$$

the dotted arrow is the unique filler induced by the factorization below. This makes K a functor and $e\colon K \longrightarrow \mathrm{id}_{\mathsf{E}}$ a natural transformation.

We define a family of co-multiplication maps d_A as the unique fillers in

$$\begin{tikzcd} KA \arrow[dr, dotted, "d_A"'] \arrow[drr, "KA"] & & \\ & KKA \arrow[r, "e_{KA}"'] & KA \end{tikzcd} \tag{22}$$

$$\begin{tikzcd} Qx \arrow[dr, "\delta_x"'] \arrow[drr, "Qx"] & & \\ & QQx \arrow[r, "\varepsilon_{Qx}"'] & Qx \end{tikzcd}$$

where the triangle below commutes because is Q co-unital.

In the diagram

$$
\begin{tikzcd}
KA \arrow[dr, "d_A"'] \arrow[drr, dotted] \arrow[drr, bend left, "KA"] \arrow[drrr, bend left, "e_A"] & & & \\
& KKA \arrow[r, shift left, "Ke_A"] \arrow[r, shift right, "e_{KA}"'] & KA \arrow[r, "e_A"'] & A
\end{tikzcd}
\tag{23}
$$

$$
\begin{tikzcd}
Qx \arrow[dr, "\delta_x"'] \arrow[drr, bend left, "Qx"] \arrow[drrr, bend left, "\varepsilon_x"] & & & \\
& QQx \arrow[r, shift left, "Q\varepsilon_x"] \arrow[r, shift right, "\varepsilon_{Qx}"'] & Qx \arrow[r, "\varepsilon_x"'] & x
\end{tikzcd}
$$

we see that $e_A e_{KA} d_A = e_A K e_A d_A$ by the naturality of e, and $p(e_{KA} d_A) = p(K e_A d_A)$ by Q being a monad. Hence by 3.10 the three endomorphisms of KA above have to coincide, meaning d is co-unital component wise.

The naturality of d, that is, that $d_B Kf = KKf d_A$ is the unique filler making the left-hand upstairs square commute

$$
\begin{tikzcd}
KA \arrow[r, "d_A"] \arrow[dr, "Kf"'] \arrow[drr, dotted] & KKA \arrow[dr, "KKf"] & & \\
& KB \arrow[r, "d_B"'] & KKB \arrow[r, "e_{KB}"'] & KB
\end{tikzcd}
\tag{24}
$$

$$
\begin{tikzcd}
Qx \arrow[r, "\delta_x"] \arrow[dr, "Qj"'] & QQx \arrow[dr, "QQj"] & & \\
& Qy \arrow[r, "\delta_y"'] & QQy \arrow[r, "\varepsilon_{Qy}"'] & Qy
\end{tikzcd}
$$

is obtained by observing that $e_{KB} d_B Kf = KF = Kf e_{KA} d_A = e_{KB} KKf d_A$, from e being natural and a retraction. Also, $p(d_B Kf) = p(KKf d_a)$ by naturality of δ. We apply 3.10 again.

Finally, we show that d is co-associative:

$$
\begin{tikzcd}
KA \arrow[r, "d_A"] \arrow[dr, "d_A"'] \arrow[drr, dotted] & KKA \arrow[dr, "d_{KA}"] & & \\
& KKA \arrow[r, "Kd_A"'] & KKKA \arrow[r, "e_{KKA}"'] & KKA
\end{tikzcd}
\tag{25}
$$

$$
\begin{tikzcd}
Qx \arrow[r, "\delta_x"] \arrow[dr, "\delta_x"'] & QQx \arrow[dr, "\delta_{Qx}"] & & \\
& QQx \arrow[r, "Q\delta_x"'] & QQQx \arrow[r, "\varepsilon_{QQx}"'] & QQx
\end{tikzcd}
$$

we calculate that $e_{KKA}Kd_Ad_A = d_Ae_{KA}d_A = d_A = e_{KKA}d_{KA}d_A$, again by naturality of e and its retractiveness. Moreover, δ is co-associative, hence we can apply 3.10 once more. $\square$

We observe that K preserves Cartesianness of maps, hence in particular Ke is Cartesian component wise.

Finally we can define our resolution comonad. Let $(Q, \delta, \varepsilon) = (FU, F\eta U, \varepsilon)$ be the comonad that arises from the adjunction

$$\begin{tikzcd} \mathsf{RGrph} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "U"] \end{tikzcd} \quad . \tag{26}$$

Then, according to theorem 3.13 we obtain the comonad (Q^1, d, e) on $\mathsf{GrayCat}$ induced by lifting Q along $(\_)_1$. The exponent reminds us that this provides a resolution of the 1-dimensional structure of Gray-categories. See B for a more abstract point of view on this construction. In section 3.18 we will show explicitly how this comonad acts.

3.14. Corollary. *By the above theorem there is a comonad* Q^1 *on* $\mathsf{GrayCat}$ *that pulls back the* Gray*-structure onto the free category on the underlying 1-graph.*

3.15. Definition. *The category of* Gray*-categories and* **pseudo Gray-maps** *is the co-Kleisli-category* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *of the comonad* Q^1*.*

This category has Gray-categories as objects, and morphisms

$$\mathbb{G} \xrightarrow{f\!\!\!/} \mathbb{H} \qquad \text{are morphisms} \qquad \mathrm{Q}^1\mathbb{G} \xrightarrow{f} \mathbb{H} \tag{27}$$

in $\mathsf{GrayCat}$. Composition of two maps

$$\mathbb{G} \xrightarrow{f\!\!\!/} \mathbb{H} \xrightarrow{g\!\!\!/} \mathbb{K} \tag{28}$$

is defined by

$$\mathrm{Q}^1\mathbb{G} \xrightarrow{d_{\mathbb{G}}} \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \xrightarrow{\mathrm{Q}^1 f} \mathrm{Q}^1\mathbb{H} \xrightarrow{g} \mathbb{K}\,. \tag{29}$$

Identities are of the form

$$\mathbb{G} \xrightarrow{\mathrm{id}_{\mathbb{G}}\!\!\!\!\!/\;\;} \mathbb{G} \;=\; \mathrm{Q}^1\mathbb{G} \xrightarrow{e_{\mathbb{G}}} \mathbb{G}\,. \tag{30}$$

By way of notational convenience in diagrams in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ we use unslashed arrows $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ to denote a strict arrow that is included in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as $fe\colon \mathbb{G} \nrightarrow \mathbb{H}$.

The comonad axioms make sure this is a category; c.f. e.g. [Mac Lane 1998].

There is an adjunction

$$\begin{tikzcd} \mathsf{GrayCat} \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \mathsf{GrayCat}_{\mathrm{Q}^1} \arrow[l, bend left, "L"] \end{tikzcd} \tag{31}$$

The functor R takes a strict map $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ to a pseudo map $fe\colon \mathbb{G} \nrightarrow \mathbb{H}$ where e is the co-unit of Q^1. Moreover, since e is an epimorphism, R is faithful, and it is bijective on objects, hence R is actually an inclusion.

We note that the composite of a strict map after a pseudo map is particularly simple:

$$
\mathbb{G} \xrightarrow{f} \mathbb{H} \xrightarrow{ge} \mathbb{K} \quad = \quad
\begin{tikzcd}
\mathrm{Q}^1\mathbb{G} \arrow[r, "d_{\mathrm{Q}^1\mathbb{G}}"] \arrow[dr, equal] & \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1 f"] \arrow[d, "e_{\mathrm{Q}^1\mathbb{G}}"] & \mathrm{Q}^1\mathbb{H} \arrow[r, "ge"] \arrow[d, "e_{\mathbb{H}}"] & \mathbb{K} \\
& \mathrm{Q}^1\mathbb{G} \arrow[r, "f"'] & \mathbb{H} \arrow[ur, "g"'] &
\end{tikzcd} . \tag{32}
$$

3.16. LEMMA. *The category* $\mathsf{GrayCat}_{\mathrm{Q}^1}$ *has all limits of diagrams of strict maps, that is, those in the subcategory* $\mathsf{GrayCat}$*, that is,* $\mathsf{GrayCat}$ *is complete and the inclusion* $\mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ *preserves all limits.*

PROOF Let D be a diagram in $\mathsf{GrayCat}$, let $(\ell_i : L \longrightarrow D_i)_i$ be a limiting source in $\mathsf{GrayCat}$, we claim its embedding into $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is a limiting source there as well.

Let $(c_i : C \twoheadrightarrow D_i)_i$ be a source over D in $\mathsf{GrayCat}_{\mathrm{Q}^1}$. Thus there is a source $(c_i : \mathrm{Q}^1 C \longrightarrow D_i)_i$ in $\mathsf{GrayCat}$, which induces a map $\langle c\rangle : \mathrm{Q}^1 C \longrightarrow L$ and this is of course a map $\langle c\rangle : C \twoheadrightarrow L$. The diagram

$$
\begin{tikzcd}
C \arrow[d, "\langle c\rangle"'] \arrow[dr, "c_i"] & \\
L \arrow[r, "\ell_i"'] & D_i
\end{tikzcd} \tag{33}
$$

commutes for all i by the co-unit axiom of Q^1 and the naturality of e; c. f. also (32). Because e is an epimorphism $\langle c\rangle$ is the unique filler. □

In particular, the pullback of two strict maps in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is the same as its pullback in $\mathsf{GrayCat}$. Products are obviously simply the same in both categories since their diagrams do not include any nontrivial morphisms.

3.17. REMARK. *For two diagrams* $\{a_k : \mathbb{G}_i \longrightarrow \mathbb{G}_j\}$, $\{b_k : \mathbb{H}_i \longrightarrow \mathbb{H}_j\}$ *of strict maps of the same type in* $\mathsf{GrayCat}_{Q^1}$ *and a natural transformation* $f_i : \mathbb{G}_i \twoheadrightarrow \mathbb{H}_i$ *between them there is an induced map* $\lim\{f_i\}$ *such that:*

$$
\begin{tikzcd}
\lim\{\mathbb{G}_i, a_k\} \arrow[r, "\lim f_i"] \arrow[d, "p_i"'] & \lim\{\mathbb{H}_i, b_k\} \arrow[d, "p'_i"] \\
\mathbb{G}_i \arrow[r, "f_i"'] & \mathbb{H}_i
\end{tikzcd} . \tag{34}
$$

We unravel this diagram in terms of maps in $\mathsf{GrayCat}$ *and obtain*

$$
\begin{tikzcd}
\mathrm{Q}^1 \lim\{\mathbb{G}_i, a_k\} \arrow[r, "\lim f_i"] \arrow[d, "\mathrm{Q}^1 p_i"'] & \lim\{\mathbb{H}_i, b_k\} \arrow[d, "p'_i"] \\
\mathrm{Q}^1\mathbb{G}_i \arrow[r, "f_i"'] & \mathbb{H}_i
\end{tikzcd} \tag{35}
$$

where the map $\lim f_i$ *is induced by the universal property of the source* $\{f_i \mathrm{Q}^1 p_i\}$ *in* $\mathsf{GrayCat}$*, that is,* $\lim\{f_i\} = \langle f_i \mathrm{Q}^1 p_i \rangle$*, which then is the appropriate map in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$*.*

In particular this applies to pullbacks, that is, there is a canonical map

$$f \times g \colon \mathbb{G} \times_{\mathbb{K}} \mathbb{H} \nrightarrow \mathbb{G}' \times_{\mathbb{K}'} \mathbb{H}' \tag{36}$$

determined by f, g, h in

$$\begin{tikzcd}
& \mathbb{H} \arrow[rr, "g"] \arrow[dd, "a"] & & \mathbb{H}' \arrow[dd, "a'"] \\
\mathbb{G} \arrow[rr, "f"] \arrow[dr, "b"'] & & \mathbb{G}' \arrow[dr, "b'"'] & \\
& \mathbb{K} \arrow[rr, "h"'] & & \mathbb{K}'
\end{tikzcd} \quad . \tag{37}$$

3.18. Special Cells in the Resolved Space. We now take a closer look at the structure of $\mathrm{Q}^1\mathbb{G}$. By definition 1-cells here are non-empty lists $[f_1, \ldots, f_n]$ of composable $\mathbb{G}$-1-cells modulo insertion or removal of identity 1-cells of $\mathbb{G}$; composition is concatenation. For composable 1-cells in $\mathbb{G}$, say, $f_1, \ldots, f_n$ we have several 1-cells in $\mathrm{Q}^1\mathbb{G}$, in particular $[f_1, \ldots, f_n] = [f_1]\#_0 \cdots \#_0[f_n]$ and $[f_1\#_0 \cdots \#_0 f_n]$ and $e_{\mathbb{G}}$ maps all of these to $f_1\#_0 \cdots \#_0 f_n$. Between $[f_1, \ldots, f_n]$ and $[f_1\#_0 \cdots \#_0 f_n]$ we have a 2-cell

$$\kappa_{f_1,\ldots,f_n} = (\mathrm{id}_{f_1\#_0\cdots\#_0 f_n}; [f_1, \ldots, f_n], [f_1\#_0 \cdots \#_0 f_n]) \tag{38}$$

that is the pulled back identity 2-cell of $f_1\#_0 \cdots \#_0 f_n$. In particular we have

$$\begin{tikzcd}
\cdot \arrow[r, "{[f_2]}"] \arrow[dr, "{[f_1\#_0 f_2]}"'] & \cdot \arrow[d, "{[f_1]}"] \arrow[dl, phantom, "\Swarrow \kappa_{f_1,f_2}"] \\
& \cdot
\end{tikzcd} \tag{39}$$

for all for all pairs f_1, f_2 of 1-cells of $\mathbb{G}$. Whiskers and composites of higher cells in $\mathrm{Q}^1\mathbb{G}$ are simply carried out in $\mathbb{G}$, hence for example

$$\begin{aligned}
\kappa_{f_1,f_2}\#_0[f_3] &= (\mathrm{id}_{f_1\#_0 f_2}\#_0 f_3; [f_1, f_2]\#_0[f_3], [f_1\#_0 f_2]\#_0[f_3]) && (40)\\
&= (\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1\#_0 f_2, f_3]) && (41)
\end{aligned}$$

and

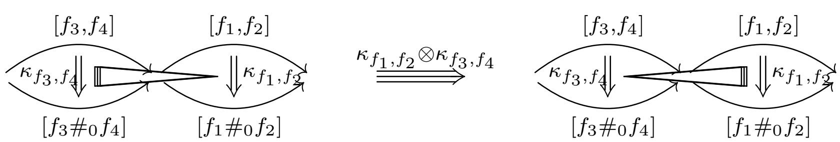
$$\kappa_{f_1\#_0 f_2, f_3}\#_1 (\kappa_{f_1,f_2}\#_0[f_3]) = (\mathrm{id}_{f_1\#_0 f_2\#_0 f_3}; [f_1, f_2, f_3], [f_1\#_0 f_2\#_0 f_3]) = \kappa_{f_1,f_2,f_3} \, . \tag{42}$$

Hence we obtain that

$$\begin{tikzcd}
{[f_1]\#_0[f_2]\#_0[f_3]} \arrow[r, Rightarrow, "{[f_1]\#_0\kappa_{f_2,f_3}}"] \arrow[d, Rightarrow, "{\kappa_{f_1,f_2}\#_0[f_3]}"'] \arrow[dr, Rightarrow, "{\kappa_{f_1,f_2,f_3}}"] & {[f_1]\#_0[f_2\#_0f_3]} \arrow[d, Rightarrow, "{\kappa_{f_1,f_2\#_0f_3}}"] \\
{[f_1\#_0f_2]\#_0[f_3]} \arrow[r, Rightarrow, "{\kappa_{f_1\#_0f_2,f_3}}"'] & {[f_1\#_0f_2\#_2f_3]}
\end{tikzcd} \tag{43}$$

commutes.

We consider the possible horizontal composites of κ_{f_1,f_2} and κ_{f_3,f_4} and their tensor:



. (44)

By (18) we obtain

$$\begin{aligned}
\kappa_{f_1,f_2}\otimes\kappa_{f_3,f_4} &= (\mathrm{id}_{f_1\#_0f_2};[f_1,f_2],[f_1\#_0f_2])\otimes(\mathrm{id}_{f_3\#_0f_4};[f_3,f_4],[f_3\#_0f_4]) \\
&= \begin{pmatrix} \mathrm{id}_{f_1\#_0f_2}\otimes\mathrm{id}_{f_3\#_0f_4}; \\ (\mathrm{id}_{f_1\#_0f_2}\#_0e[f_3\#_0f_4])\#_1(e[f_1,f_2]\#_0\mathrm{id}_{f_3\#_0f_4}), \\ (e[f_1\#_0f_2]\#_0\mathrm{id}_{f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2}\#_0e[f_3,f_4]); \\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4] \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}}; \\ (\mathrm{id}_{f_1\#_0f_2}\#_0f_3\#_0f_4)\#_1(f_1\#_0f_2\#_0\mathrm{id}_{f_3\#_0f_4}), \\ (f_1\#_0f_2\#_0\mathrm{id}_{f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2}\#_0f_3\#_0f_4); \\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4] \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}}; \\ (\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}), \\ (\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4})\#_1(\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}); \\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4] \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id}_{\mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}}; \\ \mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}, \\ \mathrm{id}_{f_1\#_0f_2\#_0f_3\#_0f_4}; \\ [f_1,f_2,f_3,f_4],[f_1\#_0f_2,f_3\#_0f_4] \end{pmatrix},
\end{aligned} \tag{45}$$

meaning that this tensor is the identity of the two possible horizontal composites of κ_{f_1,f_2} and κ_{f_3,f_4}.

Finally, note that by construction the $\kappa_{f_1,\ldots,f_n}$ are all invertible.

3.19. Pseudo Maps Explicitly. We provide an elementary characterization of pseudo Gray-functors.

3.20. Definition. *A* **pseudo Q^1 graph map** $F\colon \mathbb{G} \longrightarrow \mathbb{H}$ *between* Gray*-categories is a map of 3-globular sets, together with a function* $F^2\colon \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1 \longrightarrow \mathbb{H}_2$*, such that the following conditions hold:*

1. *the restriction of F to $\mathbb{G}(x,y)$ is a 2-functor for all 0-cells x,y of $\mathbb{G}$,*
2. *F^2 is a normalized 2-cocycle, that is, the $F^2_{f_1,f_2}$ are invertible 2-cells $F^2_{f_1,f_2}\colon F(f_1)\#_0F(f_2) \Longrightarrow F(f_1\#_0f_2)$ with*
$$F^2_{f_1,f_2\#_0f_3}\#_1(F(f_1)\#_0F^2_{f_2,f_3}) = F^2_{f_1\#_0f_2,f_3}\#_1(F^2_{f_1,f_2}\#_0F(f_3)), \tag{46}$$
and for f_1 or f_2 an identity 1-cell we have
$$F^2_{f_1,f_2} = \mathrm{id}_{f_1\#_0f_2}, \tag{47}$$
3. *left and right whiskers of 2-cells by 1-cells along 0-cells are coherently preserved:*
$$\begin{aligned} F(\alpha\#_0f)\#_1F^2_{g,f} &= F^2_{g',f}\#_1(F\alpha\#_0Ff) \\ F(g\#_0\beta)\#_1F^2_{g,f} &= F^2_{g,f'}\#_1(Fg\#_0F\beta) \end{aligned} \tag{48}$$
4. *left and right whiskers of 3-cells by 1-cells along 0-cells are coherently preserved:*
$$\begin{aligned} F(\Gamma\#_0f)\#_1F^2_{g,f} &= F^2_{g',f}\#_1(F\Gamma\#_0Ff) \\ F(g\#_0\Delta)\#_1F^2_{g,f} &= F^2_{g,f'}\#_1(Fg\#_0F\Delta) \end{aligned} \tag{49}$$
5. *the tensor is coherently preserved:*
$$F(\beta\otimes\alpha)\#_1F^2_{g,f} = F^2_{g',f'}\#_1(F\beta\otimes F\alpha) \tag{50}$$
6. *the tensors of compositors are trivial:*
$$\left(F^2_{f_1,f_2} \lhd F^2_{f_3,f_4} \xRightarrow{F^2_{f_1,f_2}\otimes F^2_{f_3,f_4}} F^2_{f_1,f_2} \rhd F^2_{fe_3,f_4} \right) = \mathrm{id} \tag{51}$$
7. *tensors of 2-co-cycle elements with images of 2-cells vanish:*
$$\left(F\alpha \lhd F^2_{g,f} \xRightarrow{F\alpha\otimes F^2_{g,f}} F\alpha \rhd F^2_{g,f} \right) = \mathrm{id} \tag{52}$$
$$\left(F^2_{h,g} \lhd F\beta \xRightarrow{F^2_{h,g}\otimes F\beta} F^2_{h,g} \rhd F\beta \right) = \mathrm{id} \tag{53}$$

for all suitably incident cells.

Note how this definition implies that the horizontal composites are also coherently preserved as a consequence of (48):

$$
\begin{aligned}
F(\alpha \triangleleft \beta) \#_1 F^2_{g,f} &= F^2_{g',f'} \#_1 (F\alpha \triangleleft F\beta) \\
F(\alpha \triangleright \beta) \#_1 F^2_{g,f} &= F^2_{g',f'} \#_1 (F\alpha \triangleright F\beta)\,.
\end{aligned} \tag{54}
$$

3.21. LEMMA. *There is a canonical correspondence between the set of pseudo* Q^1 *graph maps* $\mathbb{G} \longrightarrow \mathbb{H}$ *and* $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H})$*.*

PROOF Given a Q^1 graph map $F\colon \mathbb{G} \longrightarrow \mathbb{H}$ we define a Gray-functor $\tilde{F}\colon \mathrm{Q}^1\mathbb{G} \longrightarrow \mathbb{H}$ as follows

1. 0-cells:
$$\tilde{F}(x) = F(x), \tag{55}$$

2. 1-cells:
$$\tilde{F}[f_1, \dots, f_n] = Ff_1 \#_0 \cdots \#_0 Ff_n, \tag{56}$$

3. 2-cells:
$$\tilde{F}(\alpha; [f_1, \dots, f_n], [g_1, \dots, g_m]) = \overline{\tilde{F}\kappa_{g_1,\dots,g_m}} \#_1 F\alpha \#_1 \tilde{F}\kappa_{f_1,\dots,f_n} \tag{57}$$
where for $n = 2$ the 2-cell $\tilde{F}\kappa_{f_1,\dots,f_n}$ is defined as $F^2_{f_1,f_2}$ and for $n \geq 3$ as the unique extension due to (46) and (51),

4. 3-cells:
$$\tilde{F}(\Gamma; \alpha, \beta; [f_1, \dots, f_n], [g_1, \dots, g_m]) = \overline{\tilde{F}\kappa_{g_1,\dots,g_m}} \#_1 F\Gamma \#_1 \tilde{F}\kappa_{f_1,\dots,f_n}\,. \tag{58}$$

To elucidate, we show that 1-2-whiskers are preserved by $\tilde{F}$. For whiskerable cells

$$
\begin{tikzcd}
{} \arrow[r, "{[f_1,\dots,f_n]}"] & {} \arrow[r, bend left, "{[g_1,\dots,g_m]}"] \arrow[r, bend right, "{[g'_1,\dots,g'_{m'}]}"'] \arrow[r, phantom, "\Downarrow(\beta;\dots)"] & {}
\end{tikzcd} \tag{59}
$$

the equation

$$
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, "\tilde F[f_1,\dots,f_n]"] & \cdot \arrow[r, bend left, "\tilde F[g_1,\dots,g_m]"] \arrow[r, bend right, "\tilde F[g'_1,\dots,g'_{m'}]"'] \arrow[r, phantom, "\Downarrow \tilde F(\beta;\dots)"] & \cdot
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "Ff_1\#_0\cdots\#_0Ff_n"] & \cdot \arrow[r, bend left=80, "Fg_1\#_0\cdots\#_0Fg_m"] \arrow[r, bend left=25, "F(g_1\#_0\cdots\#_0g_m)" description] \arrow[r, bend right=25, "F(g'_1\#_0\cdots\#_0g'_{m'})" description] \arrow[r, bend right=80, "Fg'_1\#_0\cdots\#_0Fg'_{m'}"'] \arrow[r, phantom, shift left=6ex, "\Downarrow F\kappa_{g_1,\dots,g_m}"] \arrow[r, phantom, "\Downarrow F\beta"] \arrow[r, phantom, shift right=6ex, "\Downarrow \overline{F\kappa_{g'_1,\dots,g'_{m'}}}"] & \cdot
\end{tikzcd}
$$

$$
=
\begin{tikzcd}[column sep=10em, row sep=huge]
\cdot \arrow[r, bend left=80, "Fg_1\#_0\cdots\#_0Fg_m\#_0Ff_1\#_0\cdots\#_0Ff_n"] \arrow[r, bend left=25, "F(g_1\#_0\cdots\#_0g_m\#_0f_1\#_0\cdots\#_0f_n)" description] \arrow[r, bend right=25, "F(g'_1\#_0\cdots\#_0g'_{m'}\#_0f_1\#_0\cdots\#_0f_n)" description] \arrow[r, bend right=80, "Fg'_1\#_0\cdots\#_0Fg'_{m'}\#_0Ff_1\#_0\cdots\#_0Ff_n"'] \arrow[r, phantom, shift left=6ex, "\Downarrow F\kappa_{g_1,\dots,g_m,f_1,\dots,f_n}"] \arrow[r, phantom, "\Downarrow F(\beta\#_0f_1\#_0\cdots\#_0f_n)"] \arrow[r, phantom, shift right=6ex, "\Downarrow \overline{F\kappa_{g'_1,\dots,g'_{m'},f_1,\dots,f_n}}"] & \cdot
\end{tikzcd}
=
\begin{tikzcd}[column sep=10em]
\cdot \arrow[r, bend left=40, "\tilde F({[g_1,\dots,g_m]}\#_0{[f_1,\dots,f_n]})"] \arrow[r, bend right=40, "\tilde F({[g'_1,\dots,g'_{m'}]}\#_0{[f_1,\dots,f_n]})."'] \arrow[r, phantom, "\Downarrow \tilde F((\beta;\dots)\#_0{[f_1,\dots,f_n]})"] & \cdot
\end{tikzcd}
\qquad (60)
$$

is a consequence of (57). Preservation of the remaining operations is equally simple to verify.

Conversely, given a Gray-functor $G\colon \mathrm{Q}^1\mathbb{G} \longrightarrow \mathbb{H}$ we define a pseudo Q^1 graph map $\check{G}\colon \mathbb{G} \longrightarrow \mathbb{H}$ as follows:

1. 0-cells: $\check{G}(x) = G(x)$
2. 1-cells: $\check{G}(f) = G[f]$
3. 2-cells: $\check{G}(\alpha) = G(\alpha; [f], [f'])$
4. 3-cells: $\check{G}(\Gamma) = G(\Gamma; \alpha, \beta; [f], [f'])$
5. 2-co-cycle: $\check{G}^2_{f_1,f_2} = G\kappa_{f_1,f_2} = G(\mathrm{id}_{f_1\#_0 f_2}; [f_1\#_0 f_2], [f_1, f_2])$

This is obviously locally a sesquifunctor. We check the co-cycle condition:

$$
\begin{aligned}
&\check{G}^2_{f_1,f_2\#_0 f_3}\#_1(\check{G}f_1\#_0\check{G}^2_{f_2,f_3})\\
&= G(\mathrm{id}_{f_1\#_0f_2\#_0f_3}; [f_1, f_2\#_0f_3], [f_1\#_0f_2\#_0f_3])\#_1(G[f_1]\#_0G(\mathrm{id}_{f_2\#_0f_3}; [f_2, f_3], [f_2\#_0f_3]))\\
&= G(\mathrm{id}_{f_1\#_0f_2\#_0f_3}; [f_1, f_2\#_0f_3], [f_1\#_0f_2\#_0f_3])\#_1G(\mathrm{id}_{f_1\#_0f_2\#_0f_3}; [f_1, f_2, f_3], [f_1, f_2\#_0f_3])\\
&\qquad\qquad\qquad\qquad = G(id_{f_1\#_0f_2\#_0f_3}; [f_1, f_2, f_3], [f_1\#_0f_2\#_0f_3])\\
&= G(\mathrm{id}_{f_1\#_0f_2\#_0f_3}; [f_1\#_0f_2, f_3], [f_1\#_0f_2\#_0f_3])\#_1G(\mathrm{id}_{f_1\#_0f_2\#_0f_3}; [f_1, f_2, f_3], [f_1\#_0f_2, f_3])\\
&= G(\mathrm{id}_{f_1\#_0f_2\#_0f_3}; [f_1\#_0f_2, f_3], [f_1\#_0f_2\#_0f_3])\#_1(G(\mathrm{id}_{f_1\#_0f_2}; [f_1, f_2], [f_1\#_0f_2])\#_0G[f_3])\\
&\qquad\qquad\qquad\qquad\qquad\qquad = \check{G}^2_{f_1\#_0f_2,f_3}\#_1(\check{G}^2_{f_1,f_2}\#_0\check{G}f_3)
\end{aligned}
\qquad (61)
$$

Furthermore, we check the coherent preservation of whiskers:

$$\begin{aligned}
&\check{G}(\alpha\#_0 f)\#_1\check{G}^2_{g,f} \\
&\quad= G(\alpha\#_0 f;[g\#_0 f],[g'\#_0 f])\#_1 G(\mathrm{id}_{g\#_0 f};[g,f],[g\#_0 f]) \\
&\quad= G(\alpha\#_0 f;[g,f],[g'\#_0 f]) \\
&\quad= G(\mathrm{id}_{g'\#_0 f};[g',f],[g'\#_0 f])\#_1 G(\alpha\#_0;[g,f],[g',f]) \\
&\quad= G(\mathrm{id}_{g'\#_0 f};[g',f],[g'\#_0 f])\#_1(G(\alpha;[g],[g'])\#_0 G[f]) \\
&\quad= \check{G}^2_{g',f}\#_1(\check{G}\alpha\#_0\check{G}f)
\end{aligned} \tag{62}$$

The remaining axioms are verified just as easily.

We verify briefly that $\check{\check{G}} = G$, for 1-cells we have

$$\check{\check{G}}[f_1,\dots,f_n] = \check{G}f_1\#_0\dots\#_0\check{G}f_n = G[f_1]\#_0\dots\#_0 G[f_n] = G[f_1,\dots,f_n] \tag{63}$$

and for 2-cells:

$$\begin{aligned}
&\check{\check{G}}(\alpha;[f_1,\dots,f_n],[f'_1,\dots,f'_{n'}]) = \overline{\check{G}\kappa_{f'_1,\dots,f'_{n'}}}\#_1\check{G}\#_1\check{G}\kappa_{f_1,\dots,f_n} \\
&\quad= \begin{pmatrix} G(\mathrm{id}_{f'_1\#_0\cdots\#_0 f'_{n'}};[f'_1\#_0\cdots\#_0 f'_{n'}],[f'_1,\dots,f'_{n'}]) \\ \#_1 G(\alpha;[f'_1\#_0\cdots\#_0 f'_{n'}],[f_1\#_0\cdots\#_0 f_n]) \\ \#_1 G(\mathrm{id}_{f_1\#_0\cdots\#_0 f_n};[f_1,\dots,f_n],[f_1\#_0\cdots\#_0 f_n]) \end{pmatrix} \\
&\quad G(\alpha;[f_1,\dots,f_n],[f'_1,\dots,f'_{n'}])
\end{aligned} \tag{64}$$

Finally, $\check{\check{F}} = F$. □

3.22. Remark. *Given two pseudo* Q^1 *graph maps* $F\colon \mathbb{G}\longrightarrow\mathbb{H}$ *and* $G\colon\mathbb{H}\longrightarrow\mathbb{K}$ *their composite* GF *is simply the composite of the underlying globular maps with cocycle is*

$$(GF)^2_{f_1,f_2} = GF^2_{f_1,f_2}\#_1 G^2_{Ff_1,Ff_2}\,. \tag{65}$$

4. Path Spaces

We construct a path space for Gray-categories and prove some essential properties.

4.1. Definition. *Given a* Gray*-groupoid* $\mathbb{H}$ *we define the* **path space** $\overrightarrow{\mathbb{H}}$ *where the cells in each dimension are diagrams in* $\mathbb{H}$*:*

$$\overrightarrow{\mathbb{H}}_0 = \{ \xrightarrow{f} \} \tag{66}$$

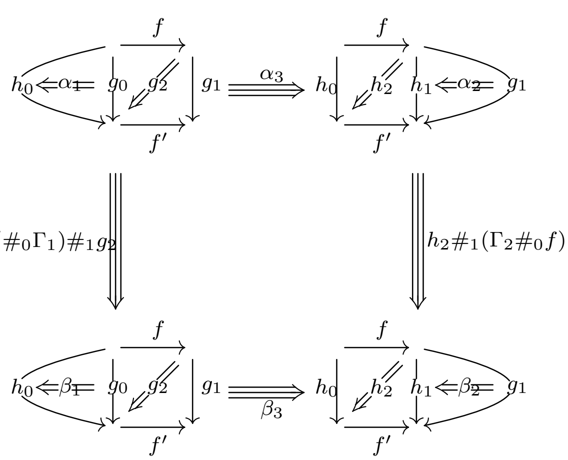
$$\overrightarrow{\mathbb{H}}_1 = \left\{ (g_2; g_0, g_1, f, f') \;\middle|\; \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \right\} \tag{67}$$

$$\overrightarrow{\mathbb{H}}_2 = \left\{ \begin{pmatrix} \alpha_3; \alpha_1, \alpha_2, g_2, h_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix} \;\middle|\; \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=90, "h_0"'] \arrow[d, phantom, bend right=45, "\Leftarrow \alpha_1"] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \overset{\alpha_3}{\Rrightarrow} \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\swarrow h_2"] & \cdot \arrow[d, "h_1"'] \arrow[d, bend left=90, "g_1"] \arrow[d, phantom, bend left=45, "\Leftarrow \alpha_2"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \right\} \tag{68}$$

$$\overrightarrow{\mathbb{H}}_3 = \left\{ \begin{pmatrix} \Gamma_1, \Gamma_2, \alpha_3, \beta_3; g_2, h_2, \\ \alpha_1, \alpha_2, \beta_1, \beta_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix} \;\middle|\; \begin{pmatrix} \Gamma_1 \colon \alpha_1 \Rrightarrow \beta_1, \\ \Gamma_2 \colon \alpha_2 \Rrightarrow \beta_2 \end{pmatrix} \text{ such that } \begin{array}{c} \beta_3 \#_2 ((f' \#_0 \Gamma_1) \#_1 g_2) \\ = (h_2' \#_1 (\Gamma_2 \#_0 f)) \#_2 \alpha_3 \end{array} \right\} \tag{69}$$

Compositions and identities arise canonically from pasting of diagrams in $\mathbb{H}$*, as detailed below.*

The condition in (69) on the 3-cells is the commutativity of the following diagram



(70)

The identities in each dimension are obviously the ones consisting of identity cells.

4.2. Remark. *By construction the map* $(d_0, d_1) \colon \overrightarrow{\mathbb{H}} \longrightarrow \mathbb{H} \times \mathbb{H}$ *is 2-faithful in the sense of definition 3.9, but in general not full.*

4.3. Remark. *The map* $i \colon \mathbb{H} \longrightarrow \overrightarrow{\mathbb{H}}$ *is 2-Cartesian and 1-faithful, but not in general 1-full.*

4.4. Path Spaces and Cartesian Maps.

4.5. LEMMA. *The path space construction* $\overrightarrow{(\_)}$ *of* Gray*-categories preserves 1-Cartesianness of maps.*

PROOF Assume a situation

$$\begin{tikzcd} \overrightarrow{\mathbb{G}} \arrow[r, "\vec{F}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & \overrightarrow{\mathbb{H}} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ \mathbb{G} \arrow[r, "F"'] & \mathbb{H} \end{tikzcd}, \tag{71}$$

assume a pair of parallel 1-cells in $\overrightarrow{\mathbb{G}}$

$$\begin{tikzcd} \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \arrow[d, "g_1"] \\ \arrow[r, "f'"'] & \end{tikzcd} \qquad\qquad \begin{tikzcd} \arrow[r, "f"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\swarrow h_2"] & \arrow[d, "h_1"] \\ \arrow[r, "f'"'] & \end{tikzcd} \tag{72}$$

we need to show that $\vec{F}$ is bijective on the intervening 2-cells. That means given

$$\beta_1 \colon F(g_0) \Longrightarrow f(h_0) \quad \beta_2 \colon F(g_1) \Longrightarrow F(h_1) \quad \beta_3 \colon F(g_2 \#_1 (\beta_2 \#_0 f)) \Rrightarrow F((f' \#_0 \beta_1) \#_1 g_2) \tag{73}$$

there are unique

$$\alpha_1 \colon g_0 \Longrightarrow h_0 \qquad \alpha_2 \colon g_1 \Longrightarrow h_1 \qquad \alpha_3 \colon g_2 \#_1 (\alpha_2 \#_0 f) \Rrightarrow (f' \#_0 \alpha_1) \#_1 g_2 \tag{74}$$

with $F(\alpha_i) = \beta_i$. But these exist uniquely by the 1-Cartesianness of F.

The same kind of argument can be applied to parallel 2-cells in $\overrightarrow{\mathbb{G}}$. $\square$

4.6. REMARK. *The functor* $\overrightarrow{(\_)}$ *preserves 2-Cartesian maps.*

4.7. LEMMA. *A pullback of a Cartesian map is Cartesian if* p *preserves pullbacks.*

PROOF Let F be p-Cartesian, and G^*F the pullback of F along G.

$$\begin{tikzcd} \cdot \arrow[dr, dashed] \arrow[drr, bend left, "\langle p(F^*G)u \rangle"] \arrow[ddr, bend right, "H"'] & & \\ & \cdot \arrow[r] \arrow[d, "G^*F"] & \cdot \arrow[d, "F"] \\ & \cdot \arrow[r, "G"'] & \cdot \end{tikzcd} \tag{75}$$

Let H factor through G below as $p(H) = p(G^*F)u$, then GH factors through F below as $p(GH) = p(GG^*F)u = p(F)p(F^*G)u$, hence there is a unique lift $\langle p(F^*G)u \rangle$. Hence there is a universally induced $\langle u \rangle$ with $G^*F\langle u \rangle = H$.

The functor p preserving pullbacks ensures that $p\langle u \rangle = u$. $\square$

4.8. Vertical Composition Operations in the Path Space. We need to describe the vertical composition of 1-, 2-, 3-cells along 0-, 1-, 2-cells respectively.

We designate the composition in $\mathbb{H}$ by $\#_i$ and the interchange by $\otimes$, in $\overrightarrow{\mathbb{H}}$ we define the respective operations $\square_i$ and $\boxtimes$ as follows:

$$h\square_0 g = (h_2; h_0, h_1, f'', f')\square_0(g_2; g_0, g_1, f, f') = \begin{pmatrix} (h_2\#_0 g_0)\#_1(h_1\#_0 g_2); \\ h_0\#_0 g_0, h_1\#_0 g_1, f, f'' \end{pmatrix} \tag{76}$$

This is just the vertical pasting

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swArrow g_2"] & \bullet \arrow[d, "g_1"] \\ \bullet \arrow[r, "f'"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\swArrow h_2"] & \bullet \arrow[d, "h_1"] \\ \bullet \arrow[r, "f''"'] & \bullet \end{tikzcd} . \tag{77}$$

Obviously this composition is associative and unital.

4.9. Remark. *Considering (77) we note that if the 1-cells in $\mathbb{H}$ are invertible, with inverse $\overline{(\_)}$, then the 2-cell*

$$(h_2\#_0 g_0)\#_1(h_1\#_0 g_2) \tag{78}$$

in (77) *can also be written as a horizontal composite in two different ways:*

$$(h_2\#_0\overline{f'}) \triangleleft g_2 = h_2 \triangleleft (\overline{f'}\#_0 g_2) \tag{79}$$

There is of course also the opposite horizontal composite

$$(h_2\#_0\overline{f'}) \triangleright g_2 = h_2 \triangleright (\overline{f'}\#_0 g_2) \tag{80}$$

and a 3-cell

$$(h_2\#_0\overline{f'})\otimes g_2 = h_2\otimes(\overline{f'}\#_0 g_2) \tag{81}$$

going from (79) *to* (80). *The picture* (77)*, however, always means* (79)*.*

The vertical composite of two 2-cells is

$$\begin{aligned} \beta\square_1\alpha = \begin{pmatrix} \beta_3; \beta_1, \beta_2, h_2, k_2; \\ h_0, h_1, k_0, k_1, f, f' \end{pmatrix} \square_1 \begin{pmatrix} \alpha_3; \alpha_1, \alpha_2, g_2, h_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix} \\ = \begin{pmatrix} (\beta_3\#_1(\alpha_2\#_0 f))\#_2((f'\#_0\beta_1)\#_1\alpha_3); \\ \beta_1\#_1\alpha_1, \beta_2\#_1\alpha_2, g_2, h_2; g_0, g_1, k_0, k_1, f, f' \end{pmatrix} \end{aligned} \tag{82}$$

which has as its first component the following composite of $\mathbb{H}$-3-cells

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=50, "h_0"'] \arrow[d, bend right=80, "k_0"'] \arrow[dr, phantom, "\swArrow g_2"] & \bullet \arrow[d, "g_1"] \\ \bullet \arrow[r, "f'"'] & \bullet \end{tikzcd} \overset{(f'\#_0\beta_1)\#_1\alpha_3}{\Rrightarrow} \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "h_0"] \arrow[d, bend right=60, "k_0"'] \arrow[dr, phantom, "\swArrow h_2"] & \bullet \arrow[d, "h_1"'] \arrow[d, bend left=60, "g_1"] \\ \bullet \arrow[r, "f'"'] & \bullet \end{tikzcd} \overset{\beta_3\#_1(\alpha_2\#_0 f)}{\Rrightarrow} \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "k_0"'] \arrow[dr, phantom, "\swArrow k_2"] & \bullet \arrow[d, "k_1"'] \arrow[d, bend left=50, "h_1"] \arrow[d, bend left=80, "g_1"] \\ \bullet \arrow[r, "f'"'] & \bullet \end{tikzcd} . \tag{83}$$

We shall henceforth argue mostly diagrammatically in terms of such 3-cell diagrams, as it is fairly obvious what the lower dimensional components are.

Vertical composition of $\overrightarrow{\mathbb{H}}$-3-cells is particularly simple:

$$\Delta\Box_2\Gamma = \begin{pmatrix}\Delta_1\colon \beta_1 \Rrightarrow \gamma_1, \\ \Delta_2\colon \beta_2 \Rrightarrow \gamma_2\end{pmatrix} \Box_2 \begin{pmatrix}\Gamma_1\colon \alpha_1 \Rrightarrow \beta_1, \\ \Gamma_2\colon \alpha_2 \Rrightarrow \beta_2\end{pmatrix} = \begin{pmatrix}\Delta_1\#_2\Gamma_1\colon \alpha_1 \Rrightarrow \gamma_1, \\ \Delta_2\#_2\Gamma_2\colon \alpha_2 \Rrightarrow \gamma_2\end{pmatrix} \tag{84}$$

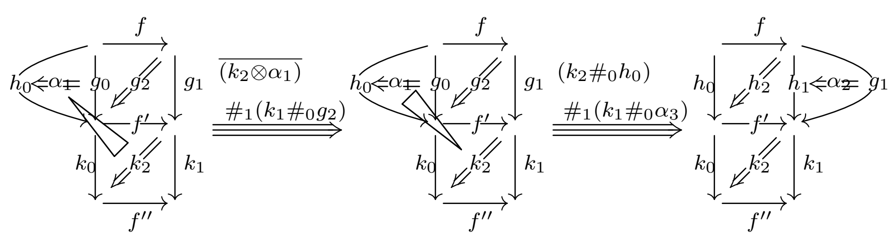
the condition 70 is obviously satisfied, since we just paste two instances of the commuting square vertically.

4.10. WHISKERS. We need to define three whiskering operations, ${}^1\Box_0^2$, ${}^1\Box_0^3$, ${}^2\Box_1^3$, where the raised indices indicate the dimension of the operands, the lower one the dimension of the incidence cell. Their symmetry partners are then obvious.

We define right whiskering of a 2-cell by a 1-cell as:

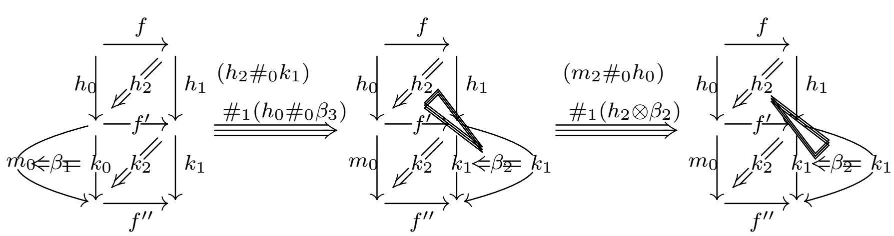
$$\begin{aligned} k\,{}^1\Box_0^2\alpha &= (k_2; k_0, k_1, f', f'')\,{}^1\Box_0^2 \begin{pmatrix}\alpha_3; \alpha_1, \alpha_2; \\ g_0, g_1, h_0, h_1, f, f'\end{pmatrix} \\ &= \begin{pmatrix} ((k_2\#_0h_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2; \\ k_0\#_0g_0, k_1\#_1g_1, k_0\#_0h_0, k_1\#_0h_1, f, f'' \end{pmatrix}. \end{aligned} \tag{85}$$

Diagrammatically this is the following composite:



(86)

For reference $(\beta_1, \beta_2, \beta_3)\Box_0(h_0, h_1, h_2)$ is



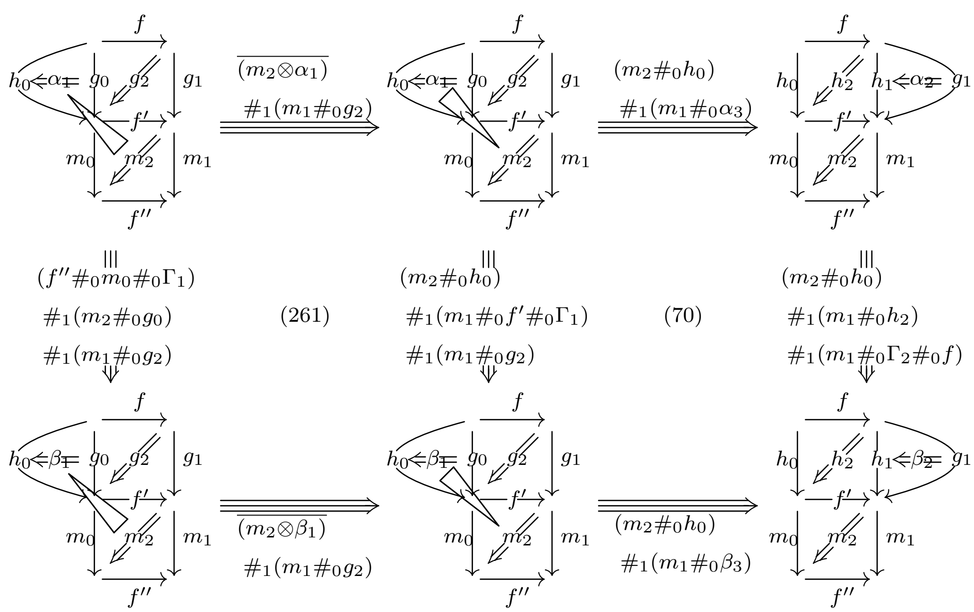
(87)

The action of 1-cells on 3-cells is as follows:

$$m^1\Box_0^3\Gamma = (m_2; m_1, m_2, f', f'')^1\Box_0^3 \begin{pmatrix} \Gamma_1, \Gamma_2, \alpha_3, \beta_3; \\ \alpha_1, \alpha_2, \beta_1\beta_2, g_2, h_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix}$$

$$= \begin{pmatrix} m_0\#_0\Gamma_1, m_1\#_0\Gamma_2, \\ ((m_2\#_0h_0)\#_1(m_1\#_0\alpha_3))\#_2(\overline{(m_2\otimes\alpha_1)}\#_1(m_1\#_0g_2)), \\ ((m_2\#_0h_0)\#_1(m_1\#_0\beta_3))\#_2(\overline{((m_2\otimes\beta_1))}\#_1(m_1\#_0g_2)); \\ m_0\#_0\alpha_1, m_0\#_1\alpha_2, m_0\#_0\beta_1, m_1\#_0\beta_2, \\ (m_2\#_0g_0)\#_1(m_1\#_0g_2), (m_2\#_0h_0)\#_1(m_1\#_0h_2); \\ m_0\#_0g_0, m_1\#_0g_1, m_0\#_0h_0, m_1\#_0h_1, f, f'' \end{pmatrix} \tag{88}$$

We claim this is again a proper 3-cell in $\overrightarrow{\mathbb{H}}$, that is, the whisker satisfies (70), as can be easily seen:



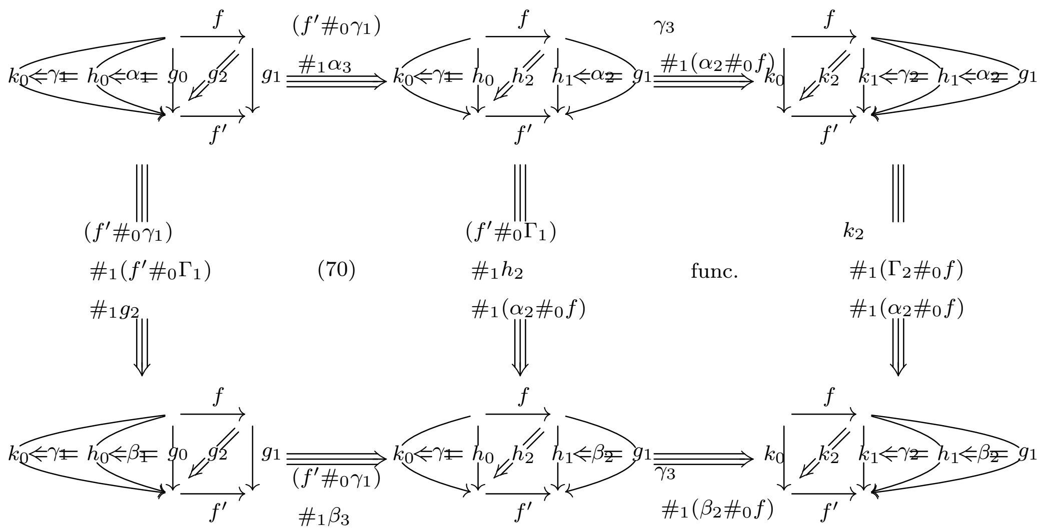
(89)

Finally, we define 3-2-whiskering:

$$\gamma^2\Box_1^3\Gamma = \begin{pmatrix} \gamma_3; \gamma_1, \gamma_2, h_2, k_2; \\ h_0, h_1, k_0, k_1, f, f' \end{pmatrix} {}^2\Box_1^3 \begin{pmatrix} \Gamma_1, \Gamma_2, \alpha_3, \beta_3; g_2, h_2, \\ \alpha_1, \alpha_2, \beta_1, \beta_2; \\ g_0, g_1, h_0, h_1, f, f' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1\#_1\Gamma_1, \gamma_2\#_1\Gamma_2, \\ (\gamma_3\#_1(\alpha_2\#_0f))\#_2((f'\#_0\gamma_1)\#_1\alpha_3), \\ (\gamma_3\#_1(\beta_2\#_0f))\#_2((f'\#_0\gamma_1)\#_1\beta_3); \\ g_2, k_2, \gamma_1\#_1\alpha_1, \gamma_2\#_1\alpha_2, \gamma_1\beta_1, \gamma_2\beta_2; \\ g_0, g_1, k_0, k_1, f, f' \end{pmatrix} \tag{90}$$

It gives a 3-cell in $\overrightarrow{\mathbb{H}}$ again.



(91)

4.11. Horizontal Composition of 2-Cells. We shall use the following slightly abbreviated notation for the higher cells of the mapping space, for example writing (85) as:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "g"] \arrow[r, bend right, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, "k"] & \cdot \end{tikzcd} = k\,{}^1\square_0^2\alpha = (k_2; k_0, k_1, f', f'')\,{}^1\square_0^2 \left(\alpha_3; \alpha_1, \alpha_2 \middle| g, n\right)$$

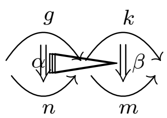
$$= \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3))\#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2 \middle| k\square_0 g, k\square_0 n \end{pmatrix} . \quad (92)$$

In the same spirit we write the opposite whiskering:

$$\begin{tikzcd} \cdot \arrow[r, "n"] & \cdot \arrow[r, bend left, "k"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\beta"] & \cdot \end{tikzcd} = \beta\,{}^2\square_0^1 n = \left(\beta_3; \beta_1, \beta_2 \middle| k, m\right)$$

$$= \begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2))\#_2(\beta_3\#_1(k_1\#_0 n_2)); \\ \beta_1\#_0 n_0, \beta_2\#_0 n_1 \middle| k\square_0 n, m\square_0 n \end{pmatrix} . \quad (93)$$

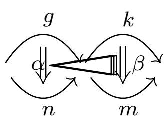
So now we can define the left horizontal composite:



$$= \beta \boxtimes\!\!\!\lhd\, \alpha = \begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)); \\ \beta_1\#_0 n_0, \beta_2\#_0 n_1 \big| k\square_0 n, m\square_0 n \end{pmatrix} \square_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2 \big| k\square_0 g, k\square_0 n \end{pmatrix}$$

$$= \begin{pmatrix} \left(\begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)) \end{pmatrix} \#_1(k_1\#_0\alpha_2\#_0 f)\right) \\ \#_2\left((f''\#_0\beta_1\#_0 n_0)\#_1\begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)) \end{pmatrix}\right); \\ \alpha_1 \lhd \beta_1, \alpha_2 \lhd \beta_2 \big| k\square_0 g, m\square_0 n \end{pmatrix} \tag{94}$$

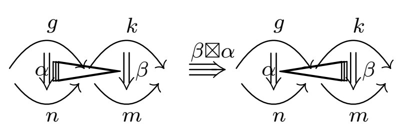
Conversely



$$= \beta \boxtimes\!\!\!\lhd\, \alpha = \begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)); \\ \beta_1\#_0 n_0, \beta_2\#_0 n_1 \big| k\square_0 n, m\square_0 n \end{pmatrix} \square_1 \begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)); \\ k_0\#_0\alpha_1, k_1\#_0\alpha_2 \big| k\square_0 g, k\square_0 n \end{pmatrix}$$

$$= \begin{pmatrix} \left(\begin{pmatrix} ((m_2\#_0 n_0)\#_1(\beta_2\otimes n_2)) \\ \#_2(\beta_3\#_1(k_1\#_0 n_2)) \end{pmatrix} \#_1(k_1\#_0\alpha_2\#_0 f)\right) \\ \#_2\left((f''\#_0\beta_1\#_0 n_0)\#_1\begin{pmatrix} ((k_2\#_0 n_0)\#_1(k_1\#_0\alpha_3)) \\ \#_2(\overline{(k_2\otimes\alpha_1)}\#_1(k_1\#_0 g_2)) \end{pmatrix}\right); \\ \alpha_1 \lhd \beta_1, \alpha_2 \lhd \beta_2 \big| k\square_0 g, m\square_0 n \end{pmatrix} \tag{95}$$

4.12. Tensors. Finally, in



(96)

letting $\beta \boxtimes \alpha = (\beta_1\otimes\alpha_1, \beta_2\otimes\alpha_2)$ makes $\overrightarrow{\mathbb{H}}$ a Gray-category. This is a well defined 3-cell.

4.13. Identities.

4.14. Inverses. If $\mathbb{H}$ has invertible 1- and 2-cells the inverse of of a 1-cell

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \tag{97}$$

in $\overrightarrow{\mathbb{H}}$ is given by

$$
\begin{tikzcd}
{} \arrow[rrr, "f'"] \arrow[dr, "\overline{g_0}"] \arrow[ddd, "\overline{g_0}"'] & & & {} \arrow[ddd, "\overline{g_1}"] \\
& {} \arrow[r, "g_0"] \arrow[d, "f"'] & {} \arrow[d, "f'"] \arrow[dl, phantom, "\Swarrow \overline{g_2}"] & \\
& {} \arrow[r, "g_1"'] & {} \arrow[dr, "\overline{g_1}"] & \\
{} \arrow[rrr, "f"'] & & & {}
\end{tikzcd} . \tag{98}
$$

4.15. Axioms. This composition of $\overrightarrow{\mathbb{H}}$-2-cells is associative: Given three 2-cells

$$
\alpha = \quad
\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, bend right=40, "\Leftarrow\alpha_1"] & {} \arrow[d, "g_1"] \arrow[dl, phantom, "\Swarrow g_2"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd}
\quad \overset{\alpha_3}{\Rrightarrow} \quad
\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "h_0"'] & {} \arrow[d, "h_1"'] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, bend left=40, "\Leftarrow\alpha_2"] \arrow[dl, phantom, "\Swarrow h_2"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd} \tag{99}
$$

$$
\beta = \quad
\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "h_0"] \arrow[d, bend right=80, "k_0"'] \arrow[d, phantom, bend right=40, "\Leftarrow\beta_1"] & {} \arrow[d, "h_1"] \arrow[dl, phantom, "\Swarrow h_2"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd}
\quad \overset{\beta_3}{\Rrightarrow} \quad
\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "k_0"'] & {} \arrow[d, "k_1"'] \arrow[d, bend left=80, "h_1"] \arrow[d, phantom, bend left=40, "\Leftarrow k_2"] \arrow[dl, phantom, "\Swarrow k_2"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd} \tag{100}
$$

$$
\gamma = \quad
\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "k_0"] \arrow[d, bend right=80, "m_0"'] \arrow[d, phantom, bend right=40, "\Leftarrow\gamma_1"] & {} \arrow[d, "k_1"] \arrow[dl, phantom, "\Swarrow k_2"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd}
\quad \overset{\gamma_3}{\Rrightarrow} \quad
\begin{tikzcd}
{} \arrow[r, "f"] \arrow[d, "m_0"'] & {} \arrow[d, "m_1"'] \arrow[d, bend left=80, "k_1"] \arrow[d, phantom, bend left=40, "\Leftarrow\gamma_2"] \arrow[dl, phantom, "\Swarrow m_2"] \\
{} \arrow[r, "f'"'] & {}
\end{tikzcd} \tag{101}
$$

we use (82) and the functoriality of the whiskerings in $\mathbb{H}$ to compute:

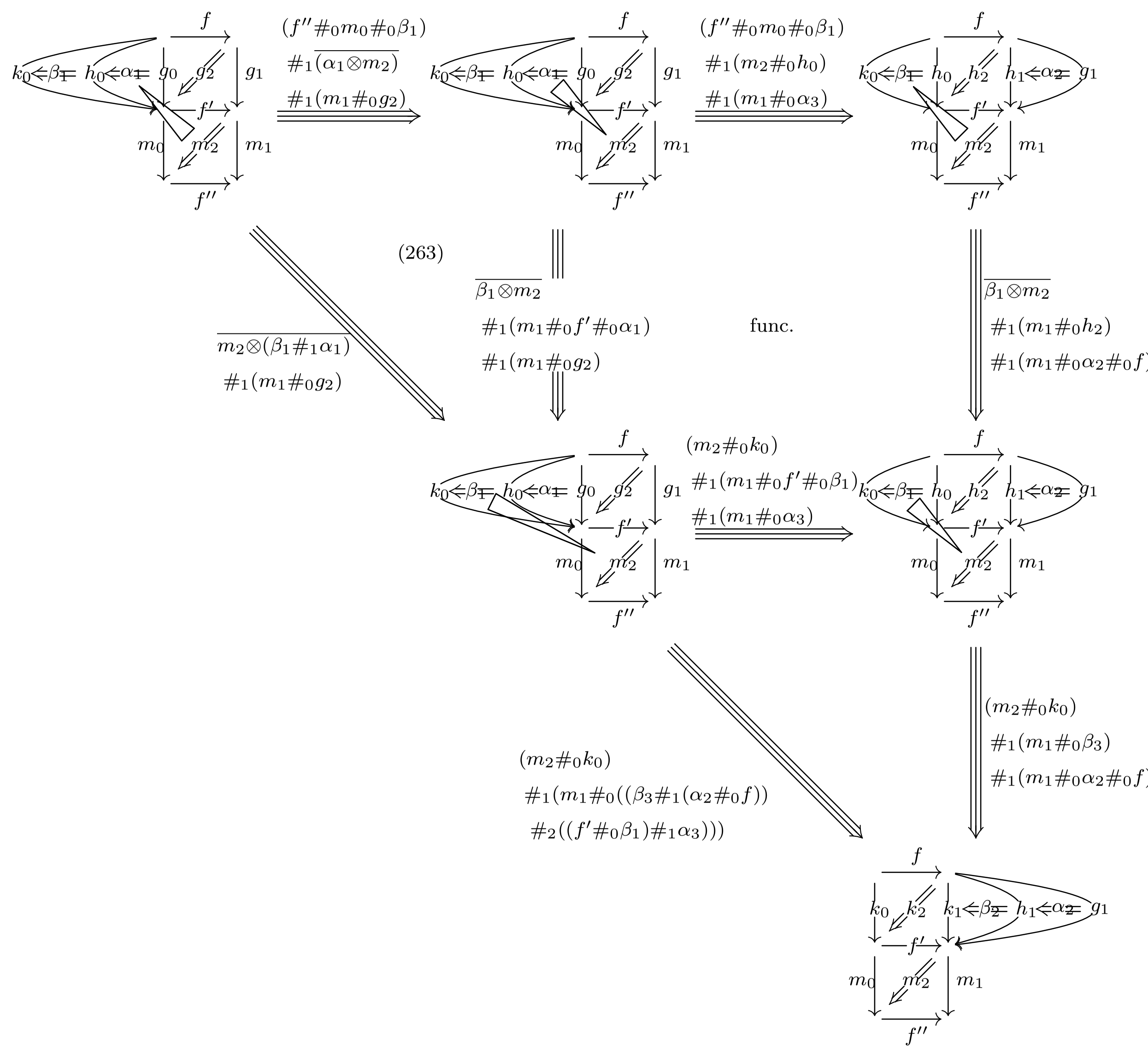
$$
\begin{aligned}
(\gamma\Box_1\beta)\Box_1\alpha &= \begin{pmatrix} \underbrace{(\gamma_3\#_1(\beta_2\#_0 f))\#_2((f'\#_0\gamma_1)\#_1\beta_3)}_{\omega_3}; \\ \gamma_1\#_1\beta_1, \gamma_2\#_1\beta_2, h_2, m_2; h_0, h_1, m_0, m_1, f, f' \end{pmatrix}\Box_1\alpha \\
&= \begin{pmatrix} (\omega_3\#_1(\alpha_2\#_0 f)) \\ \#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (((\gamma_3\#_1(\beta_2\#_0 f))\#_2((f'\#_0\gamma_1)\#_1\beta_3)) \\ \#_1(\alpha_2\#_0 f))\#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, \\ g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} = \begin{pmatrix} (\gamma_3\#_1(\beta_2\#_0 f)\#_1(\alpha_2\#_0 f)) \\ \#_2(((f'\#_0\gamma_1)\#_1\beta_3)\#_1(\alpha_2\#_0 f)) \\ \#_2((f'\#_0(\gamma_1\#_1\beta_1))\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; \\ g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0 f))\#_2((f'\#_0\gamma_1)\#_1\beta_3\#_1(\alpha_2\#_0 f)) \\ \#_2((f'\#_0\gamma_1)\#_1(f'\#_0\beta_1)\#_1\alpha_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0 f))\#_2((f'\#_0\gamma_1) \\ \#_1\underbrace{((\beta_3\#_1(\alpha_2\#_0 f))\#_2((f'\#_0\beta_1)\#_1\alpha_3))}_{\zeta_3}); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \begin{pmatrix} (\gamma_3\#_1((\beta_2\#_1\alpha_2)\#_0 f)) \\ \#_2((f'\#_0\gamma_1)\#_1\zeta_3); \\ \gamma_1\#_1\beta_1\#_1\alpha_1, \gamma_2\#_1\beta_2\#_1\alpha_2, g_2, m_2; g_0, g_1, m_0, m_1, f, f' \end{pmatrix} \\
&= \gamma\Box_1\begin{pmatrix} \zeta_3; \beta_1\#_1\alpha_1, \beta_2\#_1\alpha_2, \\ g_2, k_2; g_0, g_1, k_0, k_1, f, f' \end{pmatrix} = \gamma\Box_1(\beta\Box_1\alpha)\,. \qquad (102)
\end{aligned}
$$

We check that 2-1-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial, that is, $m\Box_0(\beta\Box_1\alpha) = (m\Box_0\beta)\Box_1(m\Box_0\alpha)$. In diagram (103) the diagonal is $m\Box_0(\beta\Box_1\alpha)$ and left and down is $(m\Box_0\beta)\Box_1(m\Box_0\alpha)$. 1-2-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial by duality.

It is obvious that 3-1-whiskering is 2-functorial, that is,

$$
\begin{aligned}
&(m_0, m_1, m_2)\Box_0((\Delta_1, \Delta_2)\Box_2(\Gamma_1, \Gamma_2)) \\
&\quad = (m_0, m_1, m_2)\Box_0(\Delta_1\#_2\Gamma_1, \Delta_2\#_2\Gamma_2) \\
&\quad = (m_0\#_0(\Delta_1\#_2\Gamma_1), m_1\#_0(\Delta_2\#_2\Gamma_2)) \\
&\quad = (((m_0\#_0\Delta_1)\#_2(m_0\#_0\Gamma_1)), ((m_1\#_0\Delta_2)\#_2(m_1\#_0\Gamma_2))) \\
&\quad = ((m_0\#_0\Delta_1), (m_1\#_0\Delta_2))\Box_2((m_0\#_0\Gamma_1), (m_1\#_0\Gamma_2)) \\
&\quad = ((m_0, m_1, m_2)\Box_0(\Delta_1, \Delta_2))\Box_2((m_0, m_1, m_2)\Box_0(\Gamma_1, \Gamma_2))\,. \qquad (104)
\end{aligned}
$$

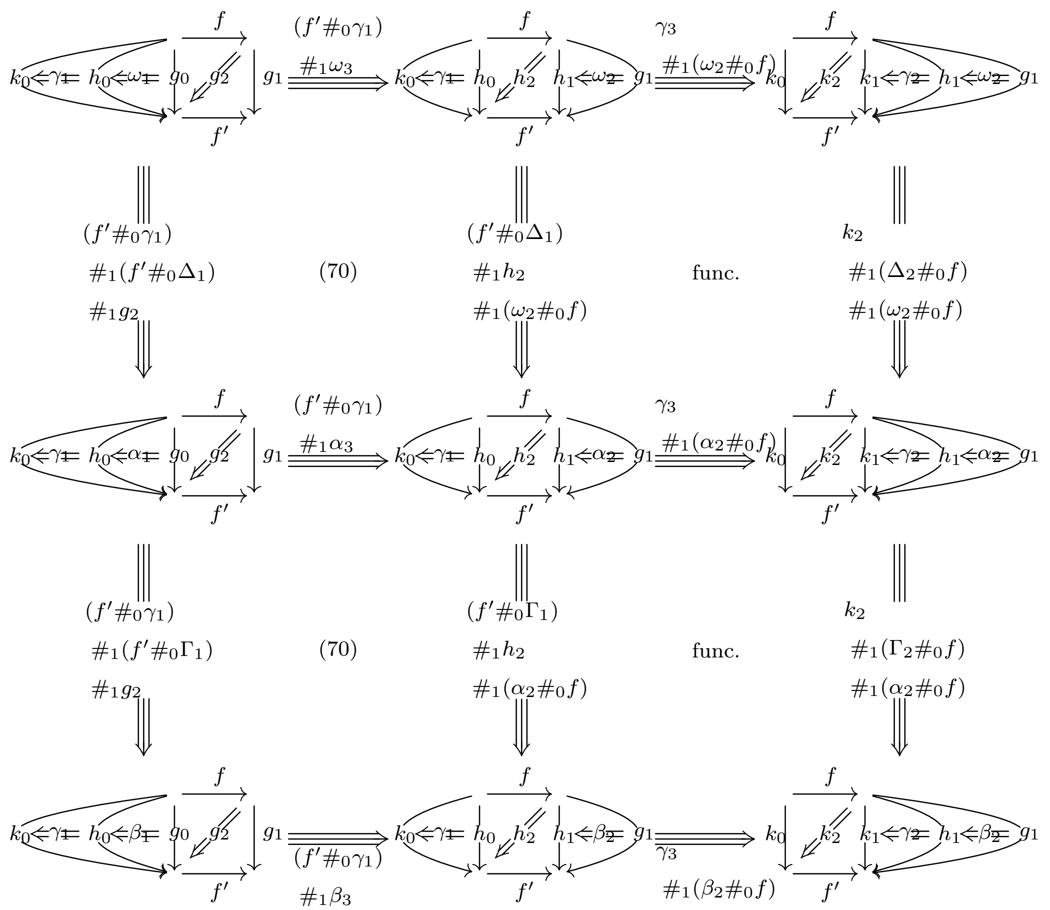
By duality, 1-2-whiskering in $\overrightarrow{\mathbb{H}}$ is functorial as well. And the 3-2-whiskering thus defined is functorial with respect to vertical composition of 3-cells, that is, $\gamma\Box_1(\Gamma\Box_2\Delta) =$



$(f''\#_0 m_0\#_0\beta_1)$
$\#_1\overline{(\alpha_1\otimes m_2)}$
$\#_1(m_1\#_0 g_2)$
$(f''\#_0 m_0\#_0\beta_1)$
$\#_1(m_2\#_0 h_0)$
$\#_1(m_1\#_0\alpha_3)$
(263)
$\overline{m_2\otimes(\beta_1\#_1\alpha_1)}$
$\#_1(m_1\#_0 g_2)$
$\overline{\beta_1\otimes m_2}$
$\#_1(m_1\#_0 f'\#_0\alpha_1)$
$\#_1(m_1\#_0 g_2)$
func.
$\overline{\beta_1\otimes m_2}$
$\#_1(m_1\#_0 h_2)$
$\#_1(m_1\#_0\alpha_2\#_0 f)$
$(m_2\#_0 k_0)$
$\#_1(m_1\#_0 f'\#_0\beta_1)$
$\#_1(m_1\#_0\alpha_3)$
$(m_2\#_0 k_0)$
$\#_1(m_1\#_0\beta_3)$
$\#_1(m_1\#_0\alpha_2\#_0 f)$
$(m_2\#_0 k_0)$
$\#_1(m_1\#_0((\beta_3\#_1(\alpha_2\#_0 f))$
$\#_2((f'\#_0\beta_1)\#_1\alpha_3)))$

(103)

$(\gamma\Box_1\Gamma)\Box_2(\gamma\Box_1\Delta)$, as can seen by inspecting the following diagram.



(105)

We see that 2-3-whiskering is functorial:

$$
\begin{aligned}
&(\Delta\Box_1\beta)\Box_2(\gamma\Box_1\Gamma)\\
&\qquad= (\Delta_1\#_1\beta_1, \Delta_2\#_1\beta_2)\Box_2(\gamma_1\#_1\Gamma_1, \gamma_2\#_1\Gamma_2)\\
&\qquad= ((\Delta_1\#_1\beta_1)\#_2(\gamma_1\#_1\Gamma_1), ((\Delta_2\#_1\beta_2)\#_2(\gamma_2\#_1\Gamma_2))\\
&\qquad= ((\delta_1\#_1\Gamma_1)\#_2(\Delta_1\#_1\alpha_1), (\delta_2\#_1\Gamma_2)\#_2(\Delta_2\#_2\alpha_2))\\
&\qquad= (\delta_1\#_1\Gamma_1, \delta_2\#_1\Gamma_2)\Box_2(\Delta_1\#_1\alpha_1, \Delta_2\#_1\alpha_2)\\
&\qquad= (\delta\Box_1\Gamma)\Box_2(\Delta\Box_1\alpha)\,. \qquad (106)
\end{aligned}
$$

So we can conclude that $\overrightarrow{\mathbb{H}}$ is locally a 2-category.

That interchange $\boxtimes$ is natural and functorial in both arguments follows immediately from the respective properties of $\otimes$ in $\mathbb{H}$. Thus we have:

4.16. LEMMA. *The path space $\overrightarrow{\mathbb{H}}$ for a* Gray*-category $\mathbb{H}$ is again a* Gray*-category.* $\Box$

4.17. Lemma. *Given a* Gray*-functor* $F\colon \mathbb{G} \longrightarrow \mathbb{H}$ *there is a canonical* Gray*-functor* $\overrightarrow{F}\colon \overrightarrow{\mathbb{G}} \longrightarrow \overrightarrow{\mathbb{H}}$.

Proof The Gray-functor $\overrightarrow{F}$ acts by applying F to all components of the cells of $\overrightarrow{\mathbb{G}}$:

$$\left(x \xrightarrow{f} y \right) \mapsto \left(Fx \xrightarrow{Ff} Fy \right) \tag{107}$$

$$\left(\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} \cdot \arrow[r, "Ff"] \arrow[d, "Fg_0"'] \arrow[dr, phantom, "\swarrow Fg_2"] & \cdot \arrow[d, "Fg_1"] \\ \cdot \arrow[r, "Ff'"'] & \cdot \end{tikzcd} \right) \tag{108}$$

$$\left(\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow \alpha_1"] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \overset{\alpha_3}{\Rrightarrow} \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\swarrow h_2"] & \cdot \arrow[d, "h_1"'] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, shift left=4ex, "\Leftarrow \alpha_2"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} \cdot \arrow[r, "Ff"] \arrow[d, "Fg_0"] \arrow[d, bend right=80, "Fh_0"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow F\alpha_1"] \arrow[dr, phantom, "\swarrow Fg_2"] & \cdot \arrow[d, "Fg_1"] \\ \cdot \arrow[r, "Ff'"'] & \cdot \end{tikzcd} \overset{F\alpha_3}{\Rrightarrow} \begin{tikzcd} \cdot \arrow[r, "Ff"] \arrow[d, "Fh_0"'] \arrow[dr, phantom, "\swarrow Fh_2"] & \cdot \arrow[d, "Fh_1"'] \arrow[d, bend left=80, "Fg_1"] \arrow[d, phantom, shift left=4ex, "\Leftarrow F\alpha_2"] \\ \cdot \arrow[r, "Ff'"'] & \cdot \end{tikzcd} \right) \tag{109}$$

$$\left(\begin{array}{ccc} \begin{tikzcd}[ampersand replacement=\&] \cdot \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow \alpha_1"] \arrow[dr, phantom, "\swarrow g_2"] \& \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] \& \cdot \end{tikzcd} & \overset{\alpha_3}{\Rrightarrow} & \begin{tikzcd}[ampersand replacement=\&] \cdot \arrow[r, "f"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\swarrow h_2"] \& \cdot \arrow[d, "h_1"'] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, shift left=4ex, "\Leftarrow \alpha_2"] \\ \cdot \arrow[r, "f'"'] \& \cdot \end{tikzcd} \\ \Big\Downarrow (f'\#_0\underline{\Gamma_1})\#_1 g_2 & & \Big\Downarrow h_2\#_1(\underline{\Gamma_2}\#_0 f) \\ \begin{tikzcd}[ampersand replacement=\&] \cdot \arrow[r, "f"] \arrow[d, "g_0"] \arrow[d, bend right=80, "h_0"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow \beta_1"] \arrow[dr, phantom, "\swarrow g_2"] \& \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] \& \cdot \end{tikzcd} & \underset{\beta_3}{\Rrightarrow} & \begin{tikzcd}[ampersand replacement=\&] \cdot \arrow[r, "f"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\swarrow h_2"] \& \cdot \arrow[d, "h_1"'] \arrow[d, bend left=80, "g_1"] \arrow[d, phantom, shift left=4ex, "\Leftarrow \beta_2"] \\ \cdot \arrow[r, "f'"'] \& \cdot \end{tikzcd} \end{array} \right) \mapsto \left(\begin{array}{ccc} \begin{tikzcd}[ampersand replacement=\&] \cdot \arrow[r, "Ff"] \arrow[d, "Fg_0"] \arrow[d, bend right=80, "Fh_0"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow F\alpha_1"] \arrow[dr, phantom, "\swarrow Fg_2"] \& \cdot \arrow[d, "Fg_1"] \\ \cdot \arrow[r, "Ff'"'] \& \cdot \end{tikzcd} & \overset{F\alpha_3}{\Rrightarrow} & \begin{tikzcd}[ampersand replacement=\&] \cdot \arrow[r, "Ff"] \arrow[d, "Fh_0"'] \arrow[dr, phantom, "\swarrow Fh_2"] \& \cdot \arrow[d, "Fh_1"'] \arrow[d, bend left=80, "Fg_1"] \arrow[d, phantom, shift left=4ex, "\Leftarrow F\alpha_2"] \\ \cdot \arrow[r, "Ff'"'] \& \cdot \end{tikzcd} \\ \Big\Downarrow (Ff'\#_0\underline{F\Gamma_1})\#_1 Fg_2 & & \Big\Downarrow Fh_2\#_1(\underline{F\Gamma_2}\#_0 Ff) \\ \begin{tikzcd}[ampersand replacement=\&] \cdot \arrow[r, "Ff"] \arrow[d, "Fg_0"] \arrow[d, bend right=80, "Fh_0"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow F\beta_1"] \arrow[dr, phantom, "\swarrow Fg_2"] \& \cdot \arrow[d, "Fg_1"] \\ \cdot \arrow[r, "Ff'"'] \& \cdot \end{tikzcd} & \underset{F\beta_3}{\Rrightarrow} & \begin{tikzcd}[ampersand replacement=\&] \cdot \arrow[r, "Ff"] \arrow[d, "Fh_0"'] \arrow[dr, phantom, "\swarrow Fh_2"] \& \cdot \arrow[d, "Fh_1"'] \arrow[d, bend left=80, "Fg_1"] \arrow[d, phantom, shift left=4ex, "\Leftarrow F\beta_2"] \\ \cdot \arrow[r, "Ff'"'] \& \cdot \end{tikzcd} \end{array} \right) \tag{110}$$

This preserves the structure of $\overrightarrow{\mathbb{G}}$ since F preserves all commting diagrams on the nose. □

4.18. Theorem. *Furthermore* $\overrightarrow{(-)}$ *is canonically an endofunctor of* GrayCat.

Proof Obviously $\overrightarrow{GF} = \overrightarrow{G}\,\overrightarrow{F}$. □

We finally note the following:

4.19. Lemma. *The functor* $\overrightarrow{(-)}$: GrayCat $\longrightarrow$ GrayCat *preserves limits.*

Proof This is obviously true for products.

For the equalizer $\mathbb{E}$ of two strict maps F, G we remember that the action of $\overrightarrow{F}$ and $\overrightarrow{G}$ is defined by the component wise action of F and G, that is, a cell of $\overrightarrow{\mathbb{E}}$ is equal under $\overrightarrow{F}$ and $\overrightarrow{G}$ iff its components are so under F and G. □

A straightforward calculation shows how this forms part of an adjunction

$$\begin{tikzcd} \mathsf{GrayCat} \arrow[r, shift left=2, "\overrightarrow{(\_)}"] \arrow[r, phantom, "\top"] & \mathsf{GrayCat} \arrow[l, shift left=2, "\_\otimes\mathbb{I}"] \end{tikzcd} \tag{111}$$

where $\mathbb{I}$ is the free Gray-category on a single 1-cell $(01)\colon 0 \longrightarrow 1$ and $\otimes$ is Crans' tensor of Gray-categories.

5. Composition of Paths

We want to turn the path space that we constructed in the previous section into the arrow part of an internal category, which requires us to define a composition map as follows.

5.1. Definition. *We define the* **composite of paths** *as a pseudo* Q^1 *graph map* $m\colon \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \twoheadrightarrow \overrightarrow{\mathbb{H}}$ *by horizontal pasting as follows:*

1. *0-cells*

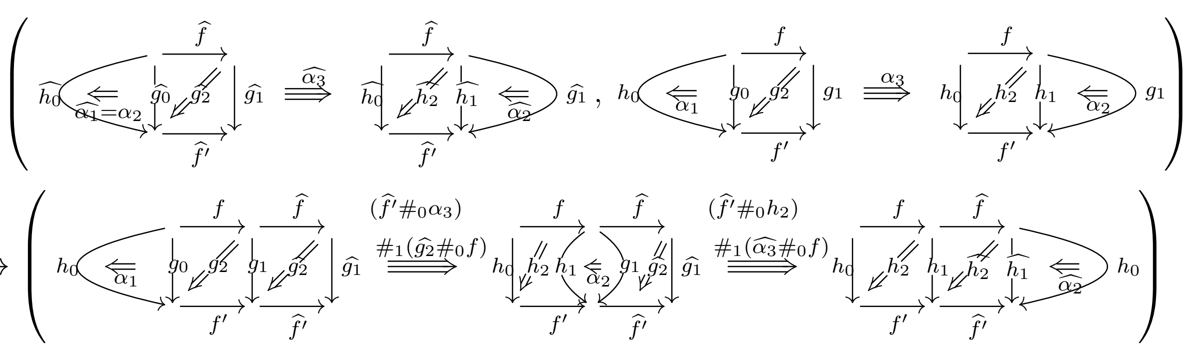
$$\left(y \xrightarrow{\widehat{f}} z\ ,\ x \xrightarrow{f} y \right) \mapsto \left(x \xrightarrow{\widehat{f}\#_0 f} z \right) \tag{112}$$

2. *1-cells*

$$\left(\begin{tikzcd} \cdot \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{g_0}=g_1"'] \arrow[dr, phantom, "\swarrow\widehat{g_2}"] & \cdot \arrow[d, "\widehat{g_1}"] \\ \cdot \arrow[r, "\widehat{f'}"'] & \cdot \end{tikzcd} ,\ \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \right) \mapsto \left(\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[r, "\widehat{f}"] \arrow[d, "g_1"] \arrow[dr, phantom, "\swarrow\widehat{g_2}"] & \cdot \arrow[d, "\widehat{g_1}"] \\ \cdot \arrow[r, "f'"'] & \cdot \arrow[r, "\widehat{f'}"'] & \cdot \end{tikzcd} \right)$$

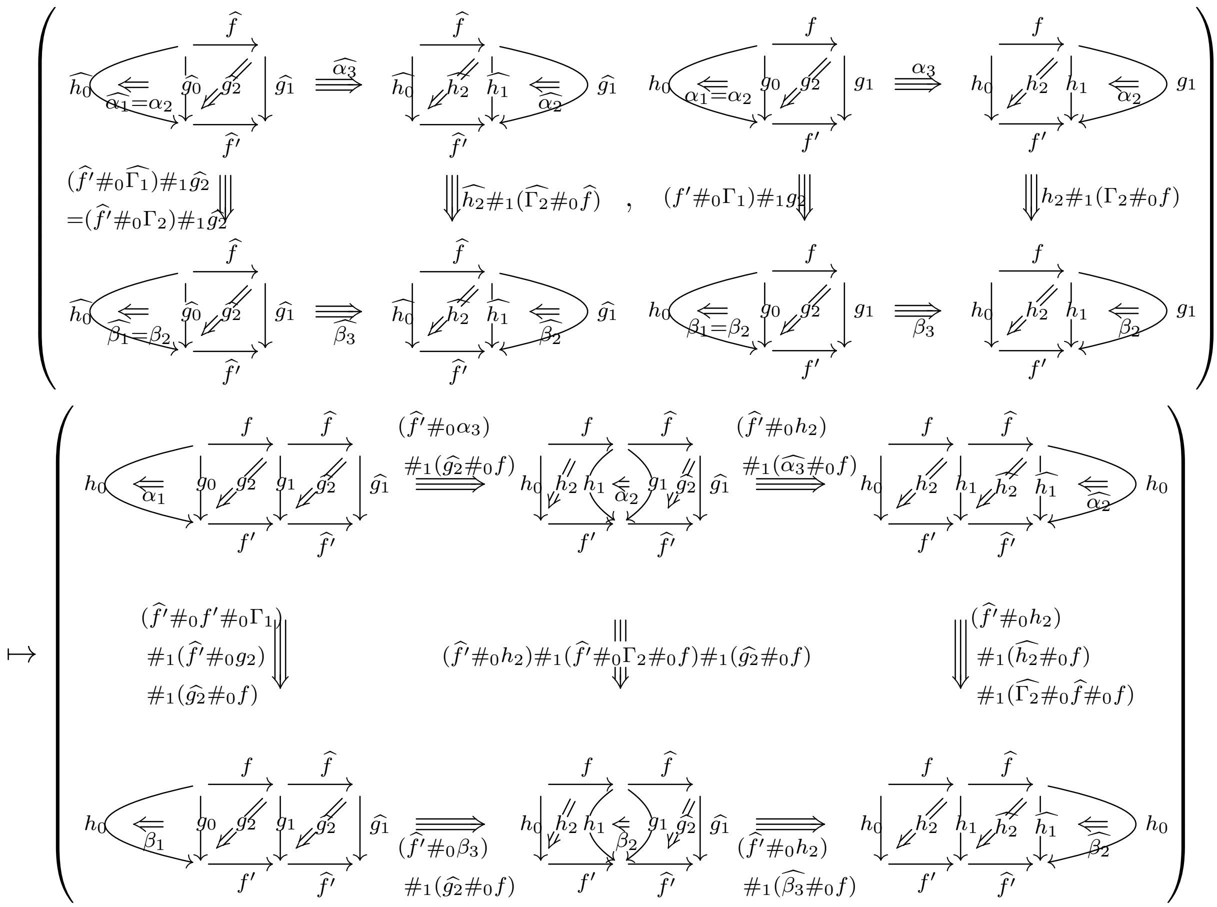
$$= \left(\begin{tikzcd} \cdot \arrow[r, "\widehat{f}\#_0 f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow (\widehat{f'}\#_0 g_2)\#_1(\widehat{g_2}\#_0 f)"] & \cdot \arrow[d, "\widehat{g_1}"] \\ \cdot \arrow[r, "\widehat{f'}\#_0 f'"'] & \cdot \end{tikzcd} \right) \tag{113}$$

3. *2-cells*



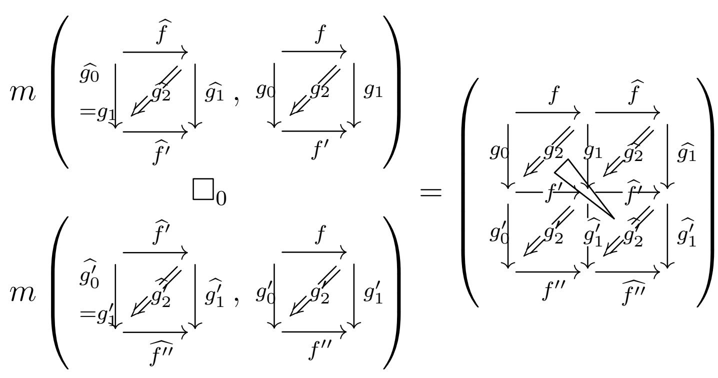
(114)

4. 3-cells

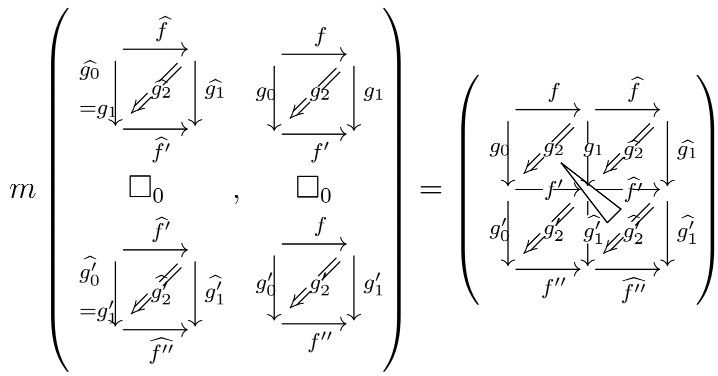


(115)

5. the 2-cocycle: for a (vertically) composable pair in $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$ we have the composite of the images and the image of the composites under m:

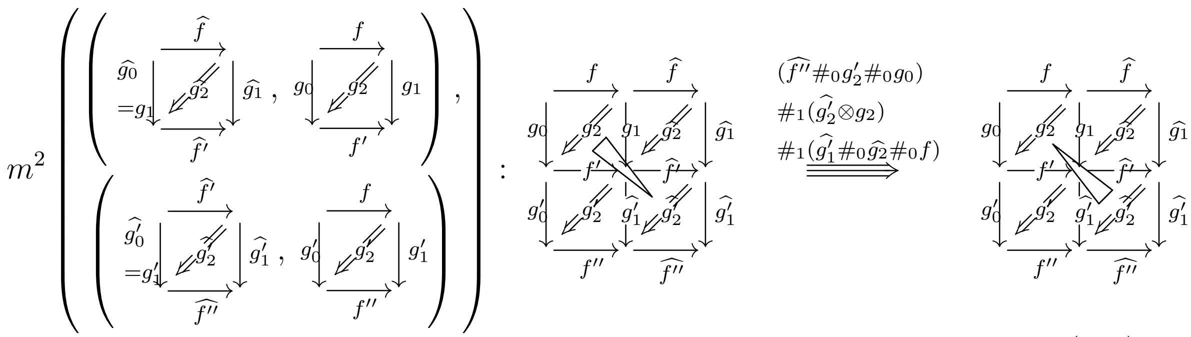


(116)



(117)

And the 2-cocycle going between them is:



(118)

For completeness' sake we give it in the algebraic notation:

$$\begin{pmatrix} (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'}\otimes g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f); \\ \mathrm{id}_{g_0'\#_0 g_0}, \mathrm{id}_{\widehat{g_1'}\#_0\widehat{g_1}}, \\ (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'}\lhd g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f), \\ (\widehat{f''}\#_0 g_2'\#_0 g_0)\#_1(\widehat{g_2'}\rhd g_2)\#_1(\widehat{g_1'}\#_0\widehat{g_2}\#_0 f); \\ g_0'\#_0 g_0, \widehat{g_1'}\#_0\widehat{g_1}, g_0'\#_0 g_0, \widehat{g_1'}\#_0\widehat{g_1}, \widehat{f}\#_0 f, \widehat{f''}\#_0 f'' \end{pmatrix} \tag{119}$$

5.2. Lemma. *The map* $m\colon \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \nrightarrow \overrightarrow{\mathbb{H}}$ *is a pseudo* Q^1 *graph map and hence by lemma 3.21 defines a pseudo* Gray*-functor.*

Proof As defined above, m is obviously a 3-globular map. We verify that it is locally a sesquifunctor: Let (β^1,β^2) and (α^1,α^2) be two pairs of 2-cells in $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$ composable along a pair of 1-cells. Then

$$m((\beta^1,\beta^2)\square_1(\alpha^1,\alpha^2)) = m((\beta^1\square_1\alpha^1),(\beta^2\square_1\alpha^2)) = m(\beta^1,\beta^2)\square_1 m(\alpha^1,\alpha^2) \tag{120}$$

follows obviously from the fact that in $\mathbb{H}$ 3-cells compose along a 2-cells interchangeably. Let (Δ^1,Δ^2) and (Γ^1,Γ^2) be two pairs of 3-cells in $\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}}$ composable along a pair of

2-cells. Then

$$
\begin{aligned}
m((\Delta^1,\Delta^2)\Box_2(\Gamma^1,\Gamma^2)) &= m((\Delta^1\Box_2\Gamma^1),(\Delta^2\Box_2\Gamma^2)) \\
&= m((\Delta^1_1\#_2\Gamma^1_1,\Delta^1_2\#_2\Gamma^1_2),(\Delta^2_1\#_2\Gamma^2_1,\Delta^2_2\#_2\Gamma^2_2)) = (\Delta^1_1\#_2\Gamma^1_1,\Delta^2_2\#_2\Gamma^2_2) \\
&= (\Delta^1_1,\Delta^2_2)\Box_2(\Gamma^1_1,\Gamma^2_2) = m((\Delta^1_1,\Delta^1_2),(\Delta^2_1,\Delta^2_2))\Box_2 m((\Gamma^1_1,\Gamma^1_2),(\Gamma^2_1,\Gamma^2_2)) \\
&= m(\Delta^1,\Delta^2)\Box_2 m(\Gamma^1,\Gamma^2)\,. \qquad (121)
\end{aligned}
$$

For the vertical composition of 3-cells see (84). Preservation of whiskers of 3-cells by 2-cells and units is equally straightforward.

We verify that m^2 is a 2-cocycle: Note that

$$
\begin{pmatrix}
(f'''^2\#_0\underline{k^1_2}\#_0h^1_0\#_0g^1_0) \\
\\
\#_1(\underline{k^2_2}\rhd\underline{h^1_2}\#_0g^1_0) \\
\#_1(k^2_1\#_0\underline{h^2_2}\rhd\underline{g^1_2}) \\
\\
\#_1(k^2_1\#_0h^2_1\#_0\underline{g^2_2}\#_0f^1)
\end{pmatrix}
=
\begin{pmatrix}
(f'''^2\#_0\underline{k^1_2}\#_0h^1_0\#_0g^1_0) \\
\#_1(f'''^2\#_0k^1_1\#_0\underline{h^1_2}\#_0g^1_0) \\
\#_1(\underline{k^2_2}\#_0h^1_1\#_0f'^1\#_0g^1_0) \\
\#_1(k^2_1\#_0f''^2\#_0h^1_1\#_0\underline{g^1_2}) \\
\#_1(k^2_1\#_0\underline{h^2_2}\#_0g^1_1\#_0f^1) \\
\#_1(k^2_1\#_0h^2_1\#_0\underline{g^2_2}\#_0f^1)
\end{pmatrix}
=
\begin{pmatrix}
\big(f'''^2\#_0((\underline{k^1_2}\#_0h^1_0) \\
\#_1(k^1_1\#_0\underline{h^1_2}))\#_0g^1_0\big) \\
\#_1(\underline{k^2_2}\lhd h^1_1\#_0\underline{g^1_2}) \\
\#_1\big(k^2_1\#_0((\underline{h^2_2}\#_0g^1_1) \\
\#_1(h^2_1\#_0\underline{g^2_2}))\#_0f^1\big)
\end{pmatrix}
\qquad (123)
$$

And the left hand rectangle in (122) commutes by local interchange. Also, m^2 is normalized by the unitality of the tensor in $\mathbb{H}$.

We check the coherent preservation of whiskers on the left of 2-cells by 1-cells in (124), where the parts commute by the naturality of the tensor and the local interchange. The corresponding condition for right whiskers is verified similarly. Coherent preservation of whiskers of 3-cells by 1-cells is a trivial consequence.

We verify the coherent preservation of tensors, i. e. that

$$
m(\beta\boxtimes\alpha)\Box_1 m^2_{k,h} = m^2_{\tilde{k},\tilde{h}}\Box_1(m(\beta)\boxtimes m(\alpha))\,, \qquad (126)
$$

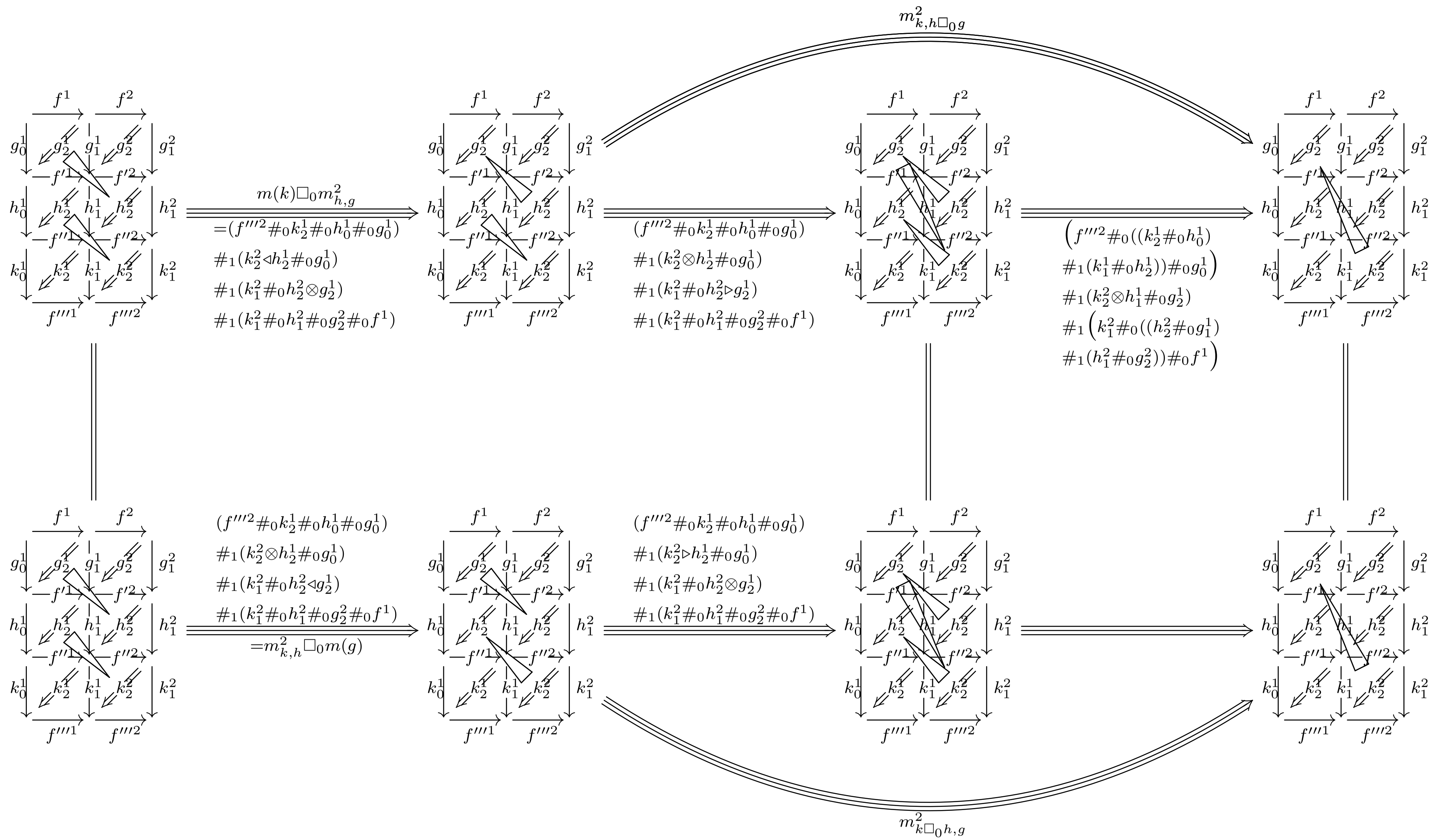
where $\alpha,\beta,k,h,\tilde{k},\tilde{h}$ are 2- and 1-cells respectively in $\overrightarrow{\mathbb{H}}\times_{\mathbb{H}}\overrightarrow{\mathbb{H}}$. In terms of constituent cells (126) can be drawn as (127), where the pasting of the center and right squares corresponds to the right hand side of the equation (126), and the pasting of the left and outer squares corresponds to the left hand side. Equality in (126) is equivalent to the top and bottom squares commuting, since the aforementioned ones do so by assumption.

We thus spell out the details of the top and bottom squares in (127): The diagram (128) shows the details of the top square of (127). The central octagon of (128) is broken down in (125). The parts of these two diagrams commute essentially by the Gray-category axioms and the definitions of 2- and 3-cells in the path space. The bottom square on (127) would be analogous.

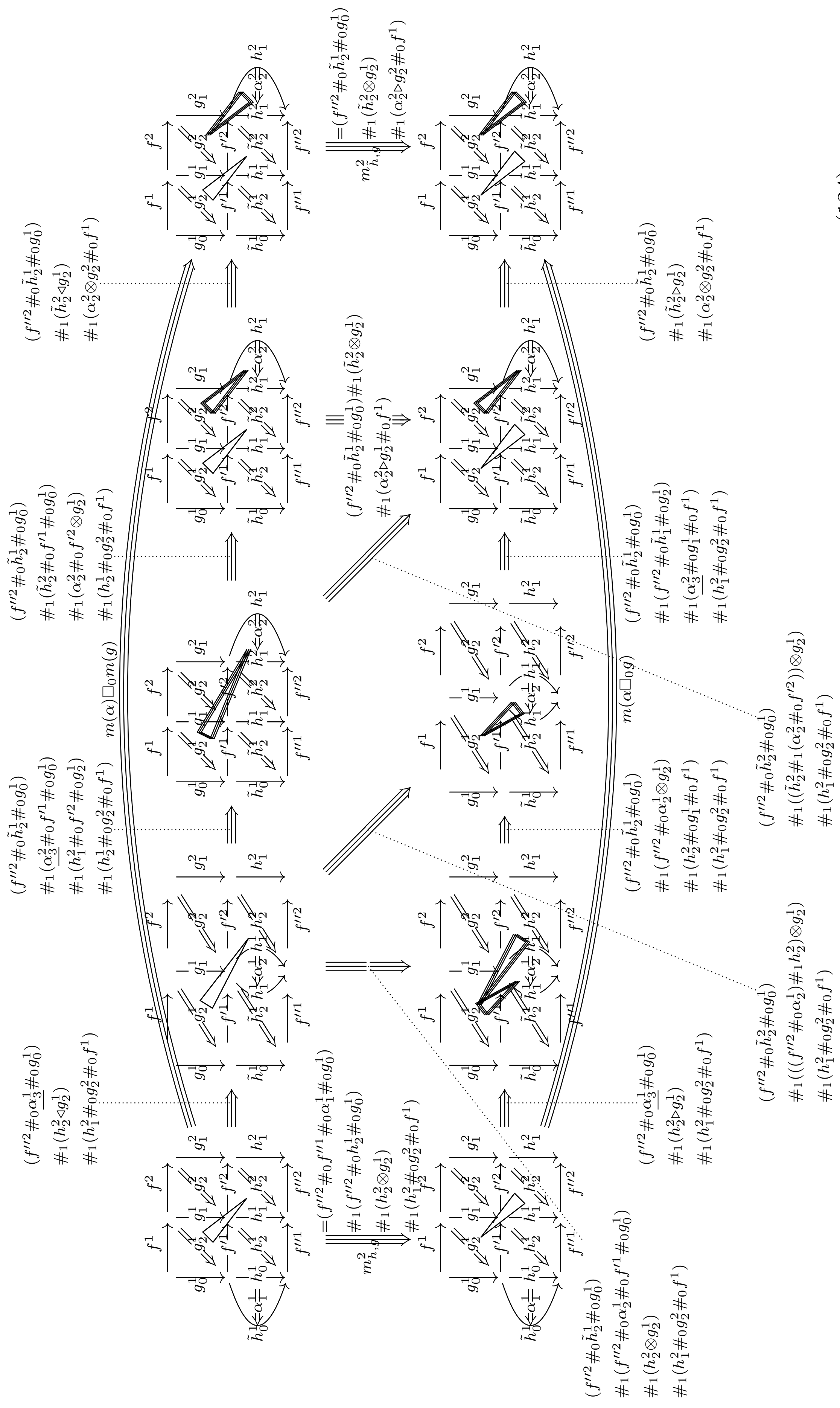
Lastly, we check that tensors of cocycle elements are trivial: We calculate according to 4.12:

$$
m^2_{f_1,f_2}\boxtimes m^2_{f_3,f_4} = ((m^2_{f_1,f_2})_1\otimes(m^2_{f_3,f_4})_1,(m^2_{f_1,f_2})_2\otimes(m^2_{f_3,f_4})_2)\,, \qquad (129)
$$

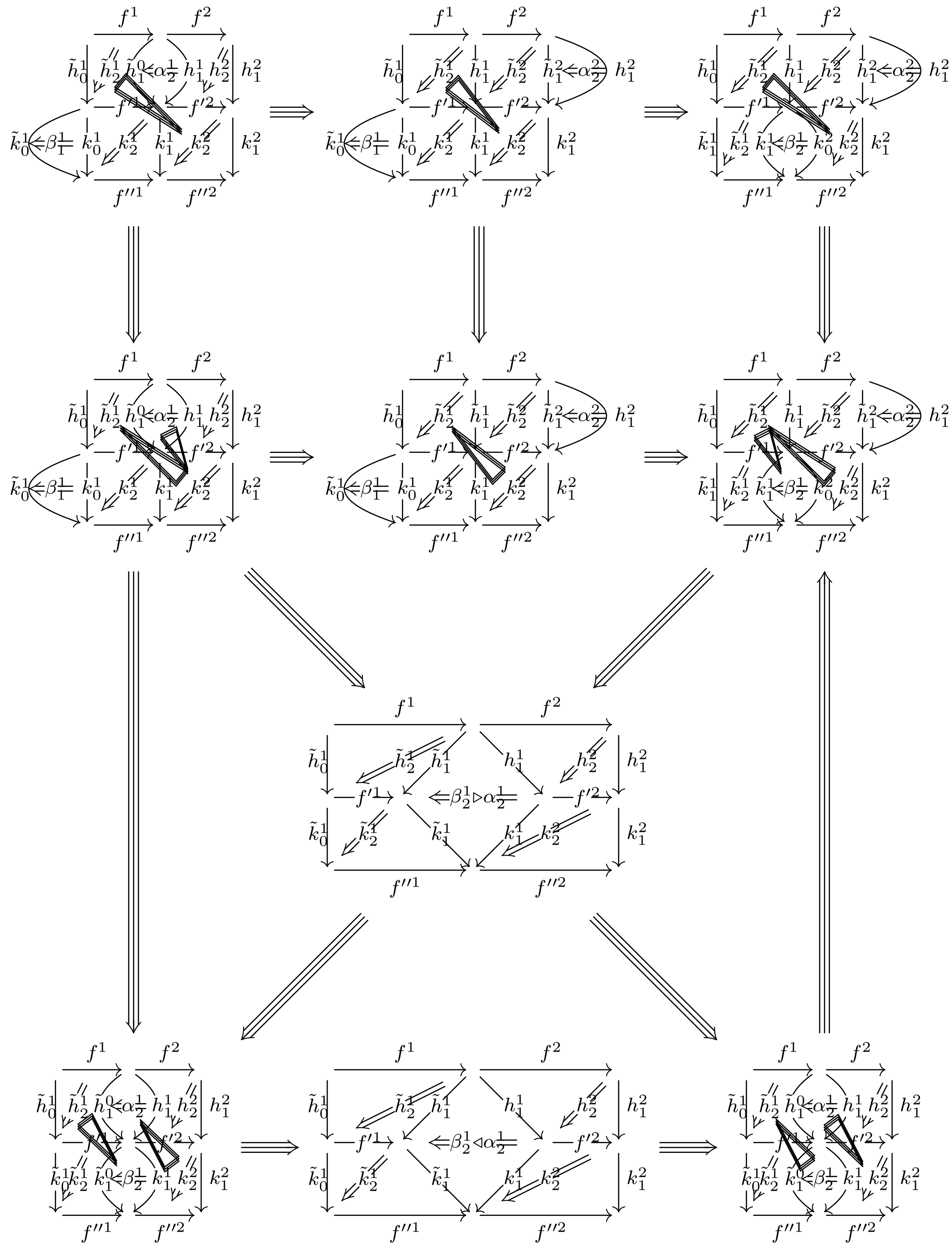
where according to (118) all the arguments on the right are trivial, hence their tensors are trivial. $\Box$


$m^2_{k,h\square_0 g}$
$m(k)\square_0 m^2_{h,g}$
$=(f'''^2\#_0 k^1_2\#_0 h^1_0\#_0 g^1_0)$
$\#_1(k^2_2\triangleleft h^1_2\#_0 g^1_0)$
$\#_1(k^2_1\#_0 h^2_2\otimes g^1_2)$
$\#_1(k^2_1\#_0 h^2_1\#_0 g^2_2\#_0 f^1)$
$(f'''^2\#_0 k^1_2\#_0 h^1_0\#_0 g^1_0)$
$\#_1(k^2_2\otimes h^1_2\#_0 g^1_0)$
$\#_1(k^2_1\#_0 h^2_2\triangleright g^1_2)$
$\#_1(k^2_1\#_0 h^2_1\#_0 g^2_2\#_0 f^1)$
$\Big(f'''^2\#_0((k^1_2\#_0 h^1_0)$
$\#_1(k^1_1\#_0 h^1_2))\#_0 g^1_0\Big)$
$\#_1(k^2_2\otimes h^1_1\#_0 g^1_2)$
$\#_1\Big(k^2_1\#_0((h^2_2\#_0 g^1_1)$
$\#_1(h^2_1\#_0 g^2_2))\#_0 f^1\Big)$
$(f'''^2\#_0 k^1_2\#_0 h^1_0\#_0 g^1_0)$
$\#_1(k^2_2\otimes h^1_2\#_0 g^1_0)$
$\#_1(k^2_1\#_0 h^2_2\triangleleft g^1_2)$
$\#_1(k^2_1\#_0 h^2_1\#_0 g^2_2\#_0 f^1)$
$=m^2_{k,h}\square_0 m(g)$
$(f'''^2\#_0 k^1_2\#_0 h^1_0\#_0 g^1_0)$
$\#_1(k^2_2\triangleright h^1_2\#_0 g^1_0)$
$\#_1(k^2_1\#_0 h^2_2\otimes g^1_2)$
$\#_1(k^2_1\#_0 h^2_1\#_0 g^2_2\#_0 f^1)$
$m^2_{k\square_0 h,g}$

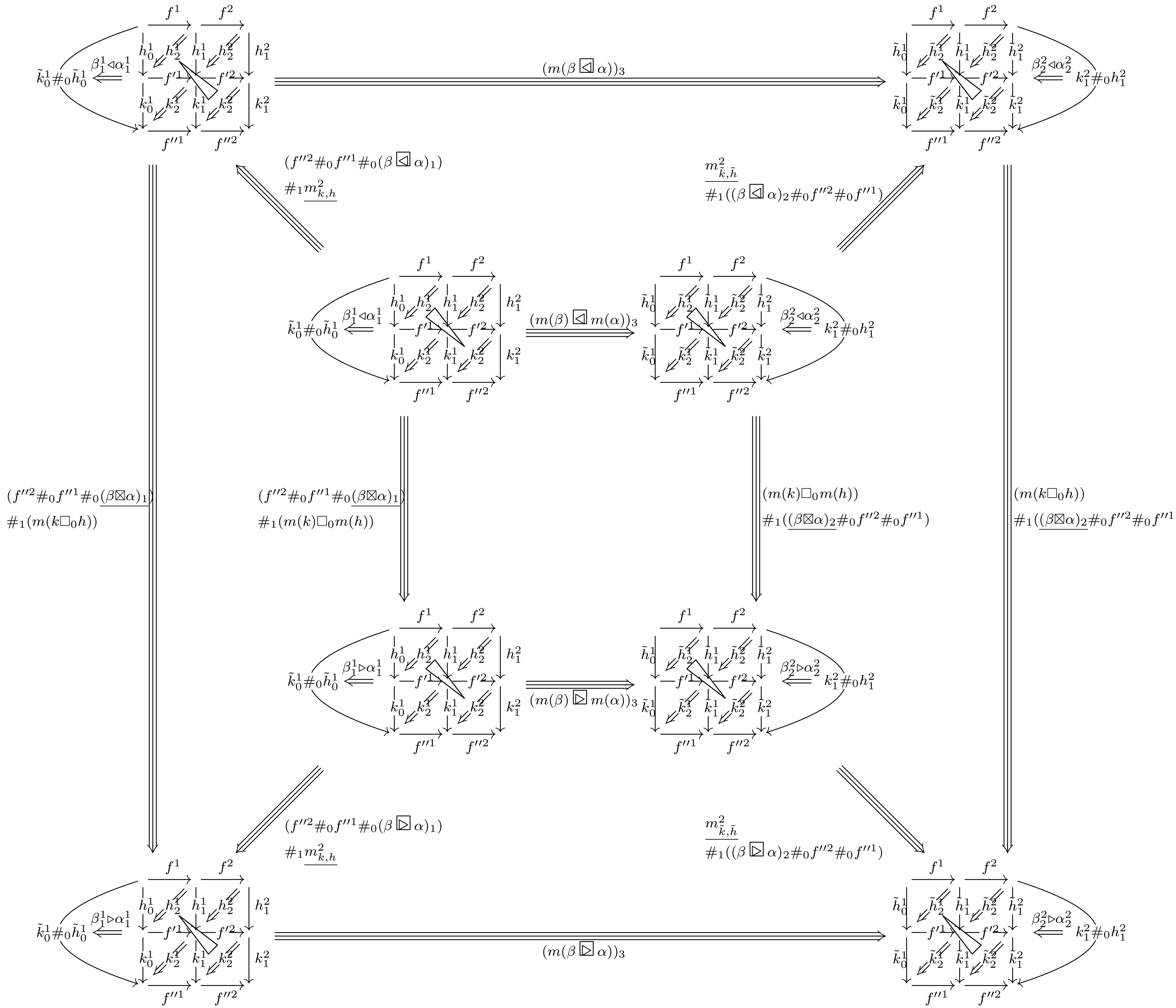
(122)



(124)

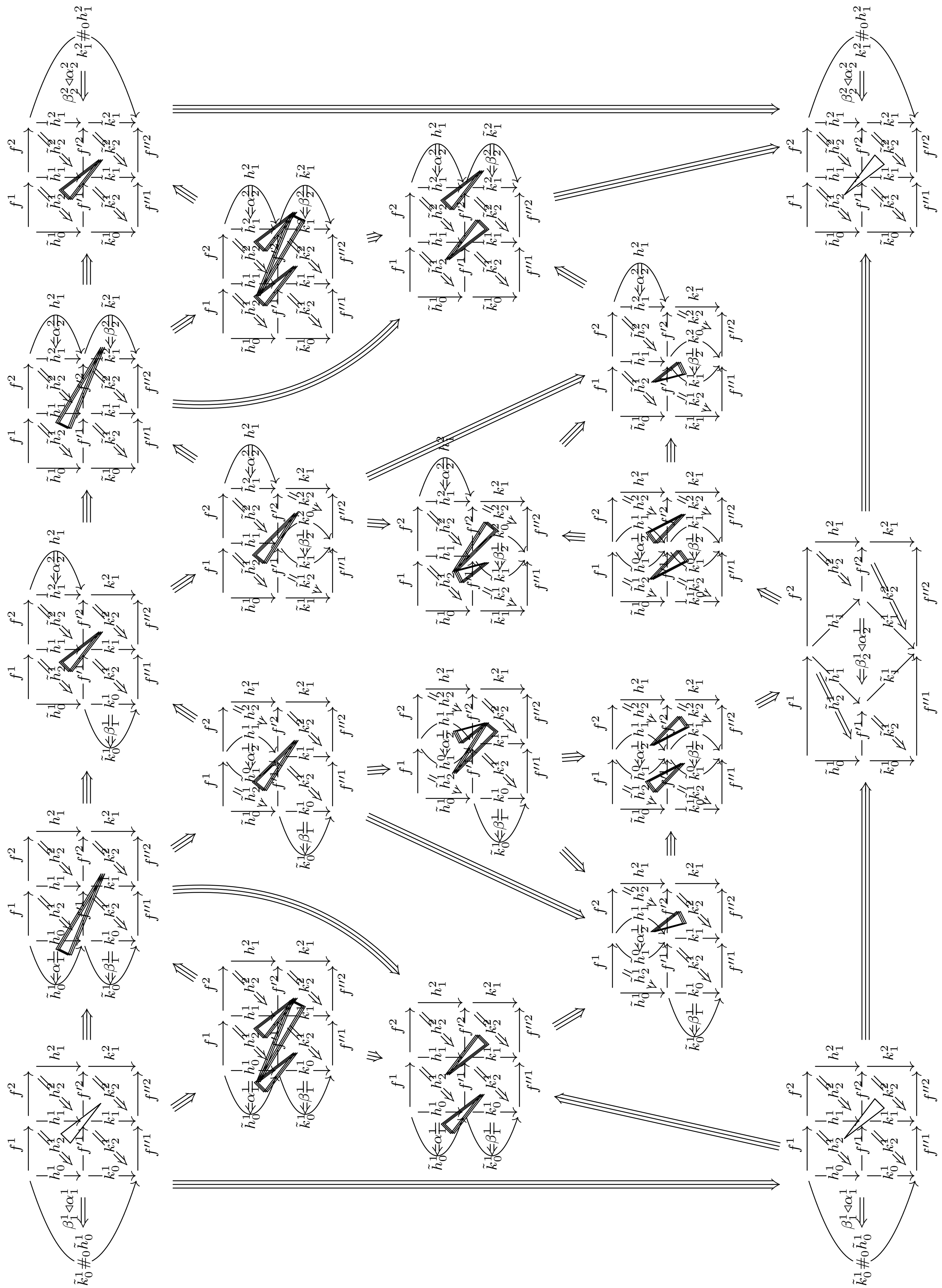


(125)



$(m(\beta \boxed{\lhd} \alpha))_3$
$(f''^2 \#_0 f''^1 \#_0 (\beta \boxed{\lhd} \alpha)_1)$
$\#_1 m^2_{k,h}$
$m^2_{\tilde{k},\tilde{h}}$
$\#_1 ((\beta \boxed{\lhd} \alpha)_2 \#_0 f''^2 \#_0 f''^1)$
$(m(\beta) \boxed{\lhd} m(\alpha))_3$
$(f''^2 \#_0 f''^1 \#_0 (\beta \boxtimes \alpha)_1)$
$\#_1 (m(k \square_0 h))$
$(f''^2 \#_0 f''^1 \#_0 (\beta \boxtimes \alpha)_1)$
$\#_1 (m(k) \square_0 m(h))$
$(m(k) \square_0 m(h))$
$\#_1 ((\beta \boxtimes \alpha)_2 \#_0 f''^2 \#_0 f''^1)$
$(m(k \square_0 h))$
$\#_1 ((\beta \boxtimes \alpha)_2 \#_0 f''^2 \#_0 f''^1)$
$(m(\beta) \boxed{\rhd} m(\alpha))_3$
$(f''^2 \#_0 f''^1 \#_0 (\beta \boxed{\rhd} \alpha)_1)$
$\#_1 m^2_{k,h}$
$m^2_{\tilde{k},\tilde{h}}$
$\#_1 ((\beta \boxed{\rhd} \alpha)_2 \#_0 f''^2 \#_0 f''^1)$
$(m(\beta \boxed{\rhd} \alpha))_3$

(127)



(128)

5.3. THEOREM. *There is a pseudo* Gray*-functor* m *such that*

$$\begin{tikzcd} \overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \arrow[r, "m", "/" marking] & \overrightarrow{\mathbb{H}} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H} \arrow[l, "i" description] \end{tikzcd} \tag{130}$$

is an internal category object in $\mathsf{GrayCat}_{\mathrm{Q}^1}$.

PROOF We need to verify that m is an associative and unital operation. The associativity of horizontal pastings is obvious, the 2-co-cycles arising from both bracketings coincide by $\mathbb{H}$ being a sesquicategory and locally a 2-category.

Unitality is obvious as well. Source and target conditions

$$\begin{tikzcd} & & \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[dl] \arrow[d, "m", "/" marking] \arrow[dr] & & \\ & \overrightarrow{\mathbb{H}} \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \overrightarrow{\mathbb{H}} \arrow[dll, "d_1"'] \arrow[drr, "d_0"] & \overrightarrow{\mathbb{H}} \arrow[dl, "d_1"'] \arrow[dr, "d_0"] & \\ \mathbb{H} & & \mathbb{H} & & \mathbb{H} \end{tikzcd} \tag{131}$$

hold by 5.1. In particular, the 2-cell components of m^2 are trivial, thus $d_{0,1}m$ are strict Gray-functors, even though m is pseudo. $\square$

We can define the 1-cell inverse to

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd} \tag{132}$$

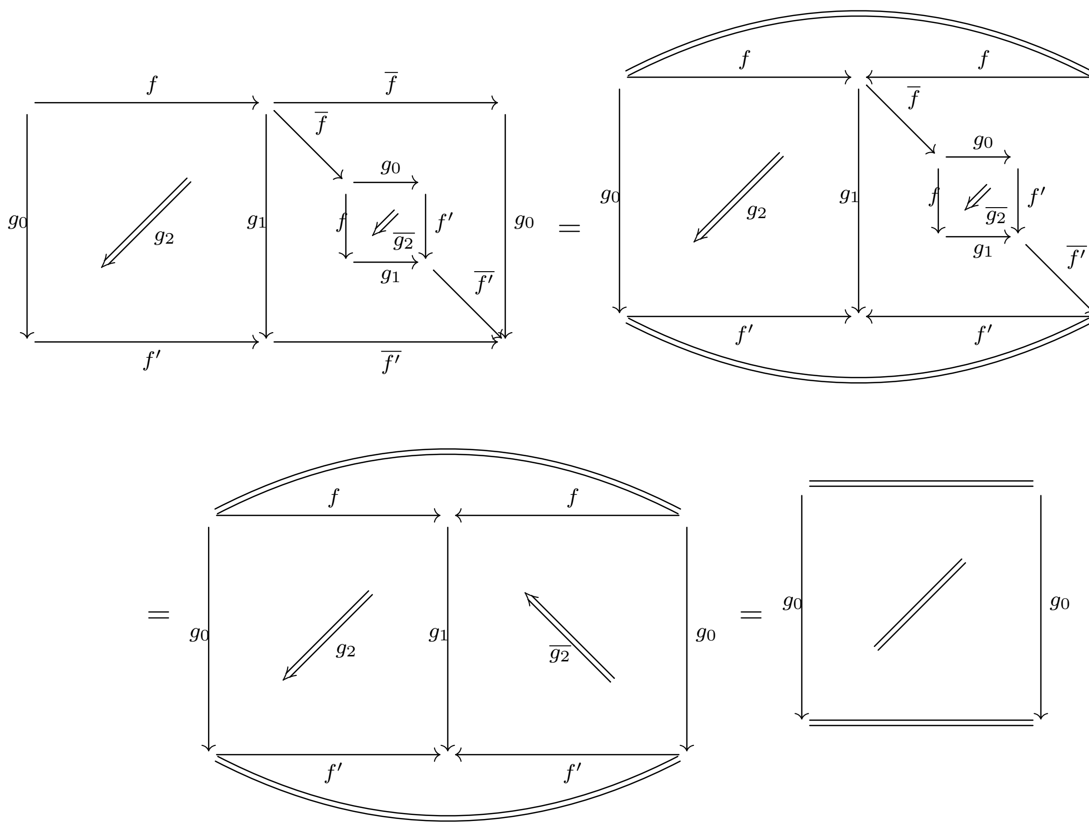
with respect to m as

$$\begin{tikzcd} \cdot \arrow[rrr, "\overline{f}"] \arrow[dddd, "g_1"'] \arrow[dr, "\overline{f}"] & & & \cdot \arrow[dddd, "g_0"] \\ & \cdot \arrow[r, "g_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \overline{g_2}"] & \cdot \arrow[d, "f'"] & \\ & \cdot \arrow[r, "g_1"'] & \cdot \arrow[ddr, "\overline{f'}"] & \\ & & & \\ \cdot \arrow[rrr, "\overline{f'}"'] & & & \cdot \end{tikzcd} \tag{133}$$

where $\overline{(\_)}$ is the respective vertical inverse in $\mathbb{H}$.

5.4. LEMMA. *The path space 1-cell in* (133) *is a left and right inverse to* (132) *with respect to* m.

Proof



(134)

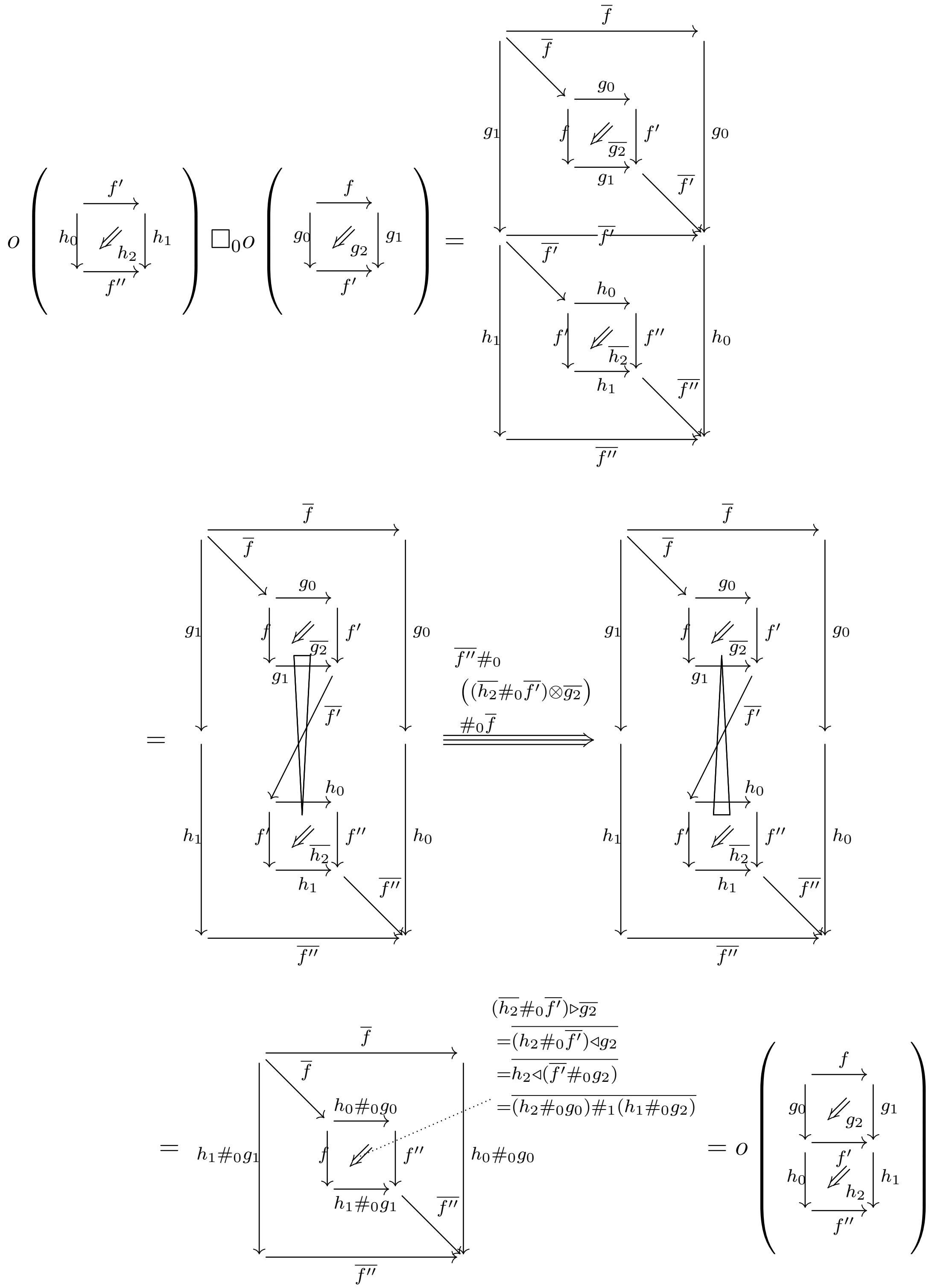
And similarly for the right inverse. $\square$

Furthermore these inverses behave well with respect to the internal category structure:

5.5. Theorem. *Given the situation in* (130)*, assume* $\mathbb{H}$ *is a* Gray*-groupoid, then there is a* Q^1*-map* $o\colon \overrightarrow{\mathbb{H}} \nrightarrow \overrightarrow{\mathbb{H}}$ *("opposite") such that* (130) *becomes an internal groupoid in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$.

Proof The action of o on 0- and 1-cells is already given in (133), we need to give its effect on 2- and 3-cells of $\overrightarrow{\mathbb{H}}$:

Furthermore, we need to give a 2-cocycle $o^2_{h,g}\colon o(h)\square_0 o(g) \longrightarrow o(h\square_0 g)$ the non-trivial part of which is the following 3-cell:



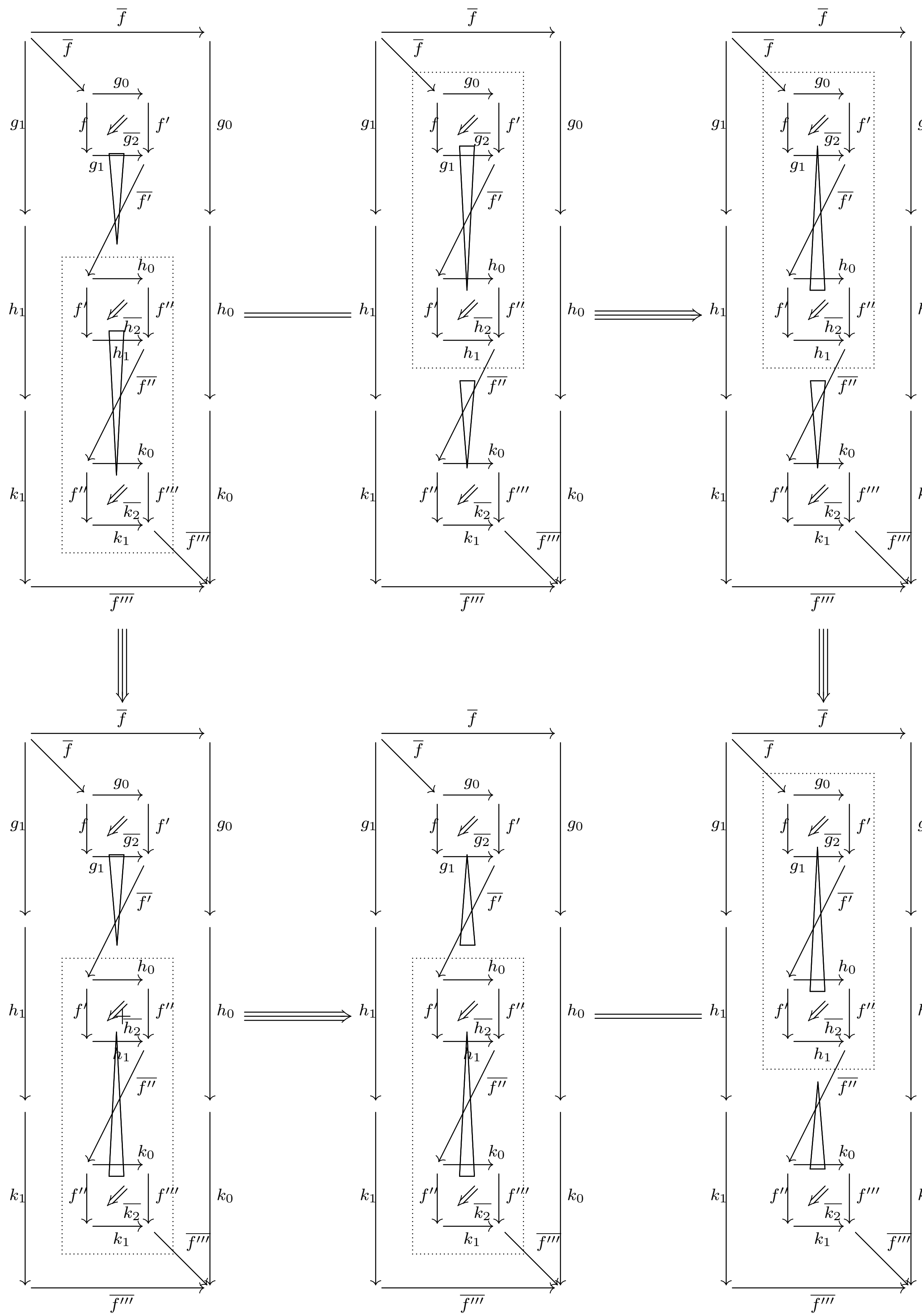
$$(135)$$

For the relationship between horizontal composition and pasting of squares see remark 4.9.

We check that o^2 is indeed a 2-cocycle. Given suitably incident 1-cells of $\mathbb{H}$ we need to verify that the analog of (46) hold, that is,

$$o^2_{k,h\Box_0 g}\Box_1(o(k)\Box_0 o^2_{h,g}) = o^2_{k\Box_0 h,g}\Box_1(o^2_{k,h}\Box_0 o(g)), \tag{136}$$

hence,



(137)

□

6. Higher Cells

In order to describe higher transformations between maps of Gray-categories we construct an internal Gray-category in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ as a substructure of the iterated path space.

6.1. Combining Path Spaces and Resolutions. We begin by describing explicitly the action of $\overrightarrow{e}\colon \overrightarrow{\mathrm{Q}^1\mathbb{G}} \longrightarrow \overrightarrow{\mathbb{G}}$ as follows:

$$\overrightarrow{e}\left(\xrightarrow{[f_1,\dots,f_{n_f}]}\right) = \left(\xrightarrow{f_1\#_0\cdots\#_0 f_{n_f}}\right) \tag{138}$$

$$\overrightarrow{e}\left(\begin{tikzcd} \cdot \arrow[r, "{[f_1,\dots,f_{n_f}]}"] \arrow[d, "{[g_{0,0},\dots,g_{0,n_{g_0}}]}"'] \arrow[dr, phantom, "{\Swarrow\,(g_2;[g_{1,0},\dots,g_{1,n_{g_1}},f_1,\dots,f_{n_f}],[f'_1,\dots,f'_{n_{f'}},g_{0,0},\dots,g_{0,n_{g_0}}])}"] & \cdot \arrow[d, "{[g_{1,0},\dots,g_{1,n_{g_1}}]}"] \\ \cdot \arrow[r, "{[f'_1,\dots,f'_{n_{f'}}]}"'] & \cdot \end{tikzcd}\right) = \left(\begin{tikzcd} \cdot \arrow[r, "f_1\#_0\cdots\#_0 f_{n_f}"] \arrow[d, "g_{0,0}\#_0\cdots\#_0 g_{0,n_{g_0}}"'] \arrow[dr, phantom, "\Swarrow g_2"] & \cdot \arrow[d, "g_{1,0}\#_0\cdots\#_0 g_{1,n_{g_1}}"] \\ \cdot \arrow[r, "f'_1\#_0\cdots\#_0 f'_{n_{f'}}"'] & \cdot \end{tikzcd}\right) \tag{139}$$

$$\overrightarrow{e}\begin{pmatrix} \begin{pmatrix}\alpha_3;[g_{1,1},\dots,g_{1,n_{g_1}},f_{1,1},\dots,f_{1,n_f}],\\ [f'_{1,1},\dots,f'_{1,n_{f'}},h_{0,1},\dots,h_{0,n_{h_0}}]\end{pmatrix};\\ (\alpha_1;[g_{0,1},\dots,g_{0,n_{g_0}}],[h_{0,1},\dots,h_{0,n_{h_0}}]),\\ (\alpha_2;[g_{1,1},\dots,g_{1,n_{g_1}}],[h_{1,1},\dots,h_{1,n_{h_1}}]),\\ \begin{pmatrix}g_2;[g_{1,1},\dots,g_{1,n_{g_1}},f_{1,1},\dots,f_{1,n_f}],\\ [f'_{1,1},\dots,f'_{1,n_{f'}},g_{0,1},\dots,g_{0,n_{h_0}}]\end{pmatrix},\\ \begin{pmatrix}h_2;[h_{1,1},\dots,h_{1,n_{h_1}},f_{1,1},\dots,f_{1,n_f}],\\ [f'_{1,1},\dots,f'_{1,n_{f'}},h_{0,1},\dots,h_{0,n_{h_0}}]\end{pmatrix};\\ [g_{0,1},\dots,g_{0,n_{g_0}}],[g_{1,1},\dots,g_{1,n_{g_1}}],\\ [h_{0,1},\dots,h_{0,n_{h_0}}],[h_{1,1},\dots,h_{1,n_{h_1}}],\\ [f_{1,1},\dots,f_{1,n_f}],[f'_{1,1},\dots,f'_{1,n_{f'}}] \end{pmatrix} = \begin{pmatrix} \alpha_3;\alpha_1,\alpha_2,g_2,h_2;\\ g_{0,1}\#_0\cdots\#_0 g_{0,n_{g_0}},g_{1,1}\#_0\cdots\#_0 g_{1,n_{g_1}},\\ h_{0,1}\#_0\cdots\#_0 h_{0,n_{h_0}},h_{1,1}\#_0\cdots\#_0 h_{1,n_{h_1}},\\ f_{1,1}\#_0\cdots\#_0 f_{1,n_f},f'_{1,1}\#_0\cdots\#_0 f'_{1,n_{f'}} \end{pmatrix} \tag{140}$$

$$\overrightarrow{e}\begin{pmatrix}(\Gamma_1;\alpha_1,\beta_1,[g_{0,1},\dots,g_{0,n_{g_0}}],[h_{0,1},\dots,h_{0,n_{h_0}}]),\\ (\Gamma_2;\alpha_2,\beta_2,[g_{1,1},\dots,g_{1,n_{g_1}}],[h_{1,1},\dots,h_{1,n_{h_1}}])\end{pmatrix} = (\Gamma_1,\Gamma_2) \tag{141}$$

where for the 3-cells we used the abbreviated notation of (69).

6.2. Lemma. *The map* $\overrightarrow{e}\colon \overrightarrow{\mathrm{Q}^1\mathbb{G}} \longrightarrow \overrightarrow{\mathbb{G}}$ *is Cartesian with respect* $(\_)_1$.

Proof $\overrightarrow{e}$ is obviously surjective on 0- and 1-cells and 2-locally an isomorphism. □

Let $F \dashv U\colon \mathsf{Cat} \longrightarrow \mathsf{RGrph}$ be the usual adjunction, then $(\overrightarrow{e})_1\colon \overrightarrow{\mathrm{Q}^1\mathbb{G}_1} \longrightarrow \overrightarrow{\mathbb{G}}_1$ has a splitting $s\colon U(\overrightarrow{\mathbb{G}}_1) \longrightarrow U(\overrightarrow{\mathrm{Q}^1\mathbb{G}_1})$ under U as follows:

$$s\left(\xrightarrow{f}\right) = \left(\xrightarrow{[f]}\right) \tag{142}$$

$$s\left(\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\Swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd}\right) = \left(\begin{tikzcd} \cdot \arrow[r, "{[f]}"] \arrow[d, "{[g_0]}"'] \arrow[dr, phantom, "{\Swarrow (g_2;[g_1,f],[f',g_0])}"] & \cdot \arrow[d, "{[g_1]}"] \\ \cdot \arrow[r, "{[f']}"'] & \cdot \end{tikzcd}\right) \tag{143}$$

Obviously in RGrph we have $U(\overrightarrow{e}_1)s = \mathrm{id}_{U(\overrightarrow{\mathbb{G}}_1)}$, taking the transpose $\overline{s}$ we get

$$\begin{tikzcd} FU(\overrightarrow{\mathbb{G}}_1) = \mathrm{Q}^1\overrightarrow{\mathbb{G}}_1 \arrow[r, "\overline{s}"] \arrow[dr, "\varepsilon = e_1"'] & \overrightarrow{\mathrm{Q}^1\mathbb{G}_1} \arrow[d, "\overrightarrow{e}_1"] \\ & \overrightarrow{\mathbb{G}}_1 \end{tikzcd}, \tag{144}$$

since $\overrightarrow{e}$ is Cartesian we can lift $\overline{s}$ through $(\_)_1$ to obtain $\psi\colon \mathrm{Q}^1\overrightarrow{\mathbb{G}} \longrightarrow \overrightarrow{\mathrm{Q}^1\mathbb{G}}$ satisfying

$$\begin{tikzcd} \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\psi_{\mathbb{G}}"] \arrow[dr, "e_{\overrightarrow{\mathbb{G}}}"'] & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[d, "\overrightarrow{e_{\mathbb{G}}}"] \\ & \overrightarrow{\mathbb{G}} \end{tikzcd}. \tag{145}$$

Let us consider the action of $\overline{s}\colon \mathrm{Q}^1\overrightarrow{\mathbb{G}}_1 \longrightarrow \overrightarrow{\mathrm{Q}^1\mathbb{G}_1}$. On 0-cells it acts just like s, on 1-cells we have the assignment:

$$\overline{s}\left(\begin{array}{c} \begin{tikzcd} \cdot \arrow[r, "f^n"] \arrow[d, "g_0^n"'] \arrow[dr, phantom, "\Swarrow g_2^n"] & \cdot \arrow[d, "g_1^n"] \\ \cdot \arrow[r, "f^{n-1}"'] & \cdot \end{tikzcd} \\ \vdots \\ \begin{tikzcd} \cdot \arrow[r, "f^1"] \arrow[d, "g_0^1"'] \arrow[dr, phantom, "\Swarrow g_2^1"] & \cdot \arrow[d, "g_1^1"] \\ \cdot \arrow[r, "f^0"'] & \cdot \end{tikzcd} \end{array}\right) = \left(\begin{tikzcd} \cdot \arrow[r, "{[f^n]}"] \arrow[d, "{[g_0^1,\dots,g_0^n]}"'] \arrow[dr, phantom, "{\Swarrow \begin{array}{l} (g_2^1\#_0 g_0^2\#_0\cdots\#_0 g_0^n) \\ \#_1\cdots \\ \#_1(g_1^1\#_0\cdots\#_0 g_2^i\#_0\cdots\#_0 g_0^n)\#_1\cdots \\ \#_1(g_1^1\#_0\cdots\#_0 g_1^{n-1}\#_0 g_2^n); \\ {[g_1^1,\dots,g_1^n,f^n],[f^0,g_0^1,\dots,g_0^n]} \end{array}}"] & \cdot \arrow[d, "{[g_1^1,\dots,g_1^n]}"] \\ \cdot \arrow[r, "{[f^0]}"'] & \cdot \end{tikzcd}\right) \tag{146}$$

6.3. Lemma. *The family ψ is natural with respect to maps $F\colon \mathbb{G} \longrightarrow \mathbb{H}$.*

Proof Consider the diagram

$$
\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\psi_{\mathbb{G}}"] \arrow[d, "\mathrm{Q}^1\overrightarrow{F}"'] \arrow[rr, bend left, "e_{\overrightarrow{\mathbb{G}}}"] & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "e_{\overrightarrow{\mathbb{G}}}"] \arrow[d, "\overrightarrow{\mathrm{Q}^1F}"'] & \overrightarrow{\mathbb{G}} \arrow[d, "\overrightarrow{F}"] \\
\mathrm{Q}^1\overrightarrow{\mathbb{H}} \arrow[r, "\psi_{\mathbb{H}}"'] \arrow[rr, bend right, "e_{\overrightarrow{\mathbb{H}}}"'] & \overrightarrow{\mathrm{Q}^1\mathbb{H}} \arrow[r, "e_{\overrightarrow{\mathbb{H}}}"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd}
, \tag{147}
$$

since the top and bottom triangles as well as the right hand square commute we obtain $e_{\overrightarrow{\mathbb{H}}}\psi_{\mathbb{H}}\mathrm{Q}^1\overrightarrow{F} = e_{\overrightarrow{\mathbb{H}}}\overrightarrow{\mathrm{Q}^1F}\psi_{\mathbb{G}}$. Since $\psi_1 = \overline{s}$ we need to only verify that $\overline{s}_{\mathbb{H}}(\mathrm{Q}^1\overrightarrow{F})_1 = (\overrightarrow{\mathrm{Q}^1F})_1\overline{s}_{\mathbb{G}}$, but this is immediate from the action of $\overrightarrow{(\_)}$ and Q^1. Naturality then follows by remark 3.10. $\square$

It remains to verify that ψ is compatible with the co-multiplication $d\colon \mathrm{Q}^1 \longrightarrow \mathrm{Q}^1\mathrm{Q}^1$, that is,

$$
\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "d_{\overrightarrow{\mathbb{G}}}"] \arrow[d, "\psi_{\mathbb{G}}"'] & \mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1\psi_{\mathbb{G}}"] & \mathrm{Q}^1\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[d, "\psi_{\mathrm{Q}^1\mathbb{G}}"] \\
\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[rr, "\overrightarrow{d_{\mathbb{G}}}"'] & & \overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}}
\end{tikzcd}
\tag{148}
$$

commutes. We will prove this using, again, remark 3.10 with $\overrightarrow{e}$ and the commutativity of the underlying diagram of categories

$$
\begin{tikzcd}
FU(\overrightarrow{\mathbb{G}}_1) \arrow[r, "F\eta U"] \arrow[d, "\overline{s}"'] & FUFU(\overrightarrow{\mathbb{G}}_1) \arrow[r, "FU\overline{s}"] & FU(\overrightarrow{\mathrm{Q}^1\mathbb{G}}_1) \arrow[d, "\overline{s}"] \\
\overrightarrow{\mathrm{Q}^1\mathbb{G}}_1 \arrow[rr, "\overrightarrow{d_{\mathbb{G}}}_1"'] & & \overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}}_1
\end{tikzcd}
\tag{149}
$$

But because the upper left object is free over the reflexive graph $U(\overrightarrow{\mathbb{G}}_1)$ it is sufficient to check for generating 0- and 1-cells.

For 0-cells we compute:

$$
\begin{aligned}
\overrightarrow{d_{\mathbb{G}}}_1\overline{s}\left(\xrightarrow{\ f\ }\right) &= \overrightarrow{d_{\mathbb{G}}}_1\left(\xrightarrow{\ [f]\ }\right) = \left(\xrightarrow{\ [[f]]\ }\right) \\
&= \overline{s}\left(\xrightarrow{\ [f]\ }\right) = \overline{s}(FU\overline{s})\left(\xrightarrow{\ f\ }\right) = \overline{s}(FU\overline{s})(F\eta U)\left(\xrightarrow{\ f\ }\right)
\end{aligned}
\tag{150}
$$

And likewise for 1-cells:

$$
\overrightarrow{d}_{\mathbb{G}1}\overline{s}\left(\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \bullet \arrow[d, "g_1"] \\ \bullet \arrow[r, "f'"'] & \bullet \end{tikzcd}\right) = \overrightarrow{d}_{\mathbb{G}1}\left(\begin{tikzcd} \bullet \arrow[r, "{[f]}"] \arrow[d, "{[g_0]}"'] \arrow[dr, phantom, "{(g_2;[g_1,f],[f',g_0])}"] & \bullet \arrow[d, "{[g_1]}"] \\ \bullet \arrow[r, "{[f']}"'] & \bullet \end{tikzcd}\right) = \left(\begin{tikzcd} \bullet \arrow[r, "{[[f]]}"] \arrow[d, "{[[g_0]]}"'] \arrow[dr, phantom, "{(g_2;[[g_1],[f]],[[f]',[g_0]])}"] & \bullet \arrow[d, "{[[g_1]]}"] \\ \bullet \arrow[r, "{[[f']]}"'] & \bullet \end{tikzcd}\right)
$$

$$
= \overline{s}\left(\begin{tikzcd} \bullet \arrow[r, "{[f]}"] \arrow[d, "{[g_0]}"'] \arrow[dr, phantom, "{(g_2;[g_1,f],[f',g_0])}"] & \bullet \arrow[d, "{[g_1]}"] \\ \bullet \arrow[r, "{[f']}"'] & \bullet \end{tikzcd}\right) = \overline{s}(FU\overline{s})\left(\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \bullet \arrow[d, "g_1"] \\ \bullet \arrow[r, "f'"'] & \bullet \end{tikzcd}\right) = \overline{s}(FU\overline{s})(F\eta U)\left(\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \bullet \arrow[d, "g_1"] \\ \bullet \arrow[r, "f'"'] & \bullet \end{tikzcd}\right) \tag{151}
$$

Furthermore we can check that post-composing (148) with $\overrightarrow{e}$ gives a commuting diagram:

$$
\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "d_{\overrightarrow{\mathbb{G}}}"] \arrow[d, "\psi_{\mathbb{G}}"'] \arrow[dr, "\mathrm{Q}^1\overrightarrow{\mathbb{G}}"'] & \mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1\psi_{\mathbb{G}}"] \arrow[d, "e_{\mathrm{Q}^1\overrightarrow{\mathbb{G}}}"] & \mathrm{Q}^1\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "\psi_{\mathrm{Q}^1\mathbb{G}}"] \arrow[ddr, "e_{\overrightarrow{\mathrm{Q}^1\mathbb{G}}}"'] & \overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}} \arrow[dd, "\overrightarrow{e_{\mathrm{Q}^1\mathbb{G}}}"] \\
\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[d, "\overrightarrow{d_{\mathbb{G}}}"'] \arrow[drrr, "\overrightarrow{\mathrm{Q}^1\mathbb{G}}"'] & \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[drr, "\psi_{\mathbb{G}}"] & & \\
\overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}} \arrow[rrr, "\overrightarrow{e_{\mathrm{Q}^1\mathbb{G}}}"'] & & & \overrightarrow{\mathrm{Q}^1\mathbb{G}}
\end{tikzcd} \tag{152}
$$

using (145), naturality of ψ in 6.3, and the fact that Q^1 is a comonad. Hence we can cancel $\overrightarrow{e}$ and obtain (148).

So, we have proved the following

6.4. Lemma. *There is a natural transformation $\psi\colon \mathrm{Q}^1\overrightarrow{(\_)} \longrightarrow \overrightarrow{\mathrm{Q}^1(\_)}$ satisfying properties* (145) *and* (148)*. We call it a* **semi-distributive law**. □

6.5. Lemma. *The functor $\overrightarrow{(\_)}$ extends canonically to an endofunctor $\wp$ of $\mathsf{GrayCat}_{\mathrm{Q}^1}$ by*

$$
\wp\left(\mathbb{G} \overset{f}{\nrightarrow} \mathbb{H}\right) = \left(\mathrm{Q}^1\overrightarrow{\mathbb{G}} \xrightarrow{\psi} \overrightarrow{\mathrm{Q}^1\mathbb{G}} \xrightarrow{\overrightarrow{f}} \overrightarrow{\mathbb{H}}\right) = \left(\overrightarrow{\mathbb{G}} \overset{\wp(f)}{\nrightarrow} \overrightarrow{\mathbb{H}}\right). \tag{153}
$$

Furthermore, it preserves strict maps as well as limits of diagrams of strict maps.

Proof We use the properties of ψ to check that this assignment is functorial. Given two maps $f\colon \mathbb{G} \nrightarrow \mathbb{H}$ and $g\colon \mathbb{H} \nrightarrow \mathbb{K}$ we compare $\wp(g)\wp(f)$ at the top and $\wp(gf)$ at the

bottom:

$$
\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "d"] \arrow[dr, "\psi"'] & \mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1\psi"] & \mathrm{Q}^1\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "\mathrm{Q}^1\overrightarrow{f}"] \arrow[d, "\psi"'] & \mathrm{Q}^1\overrightarrow{\mathbb{H}} \arrow[r, "\psi"] & \overrightarrow{\mathrm{Q}^1\mathbb{H}} \arrow[r, "\overrightarrow{g}"] & \overrightarrow{\mathbb{K}} \\
& \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "\overrightarrow{d}"'] & \overrightarrow{\mathrm{Q}^1\mathrm{Q}^1\mathbb{G}} \arrow[urr, "\overrightarrow{\mathrm{Q}^1 f}"'] & & &
\end{tikzcd} . \tag{154}
$$

The naturality of ψ and (148) make sure they are equal. Preservation of units is exactly (145).

We remember that a strict map in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ is given by $fe_{\mathbb{G}}$ where $f\colon \mathbb{G} \longrightarrow \mathbb{H}$ is from $\mathsf{GrayCat}$ and e is the co-unit of Q^1. Then by (145) we get

$$
\wp(fe_{\mathbb{G}}) = \overrightarrow{f}\,\overrightarrow{e_{\mathbb{G}}}\psi_{\mathbb{G}} = \overrightarrow{f}\,e_{\overrightarrow{\mathbb{G}}} \,, \tag{155}
$$

Meaning that $\wp$ acts on strict maps like $\overrightarrow{(\_)}$, in particular, it takes identities to identities.

Finally, by lemma 4.19 the restriction of $\overrightarrow{(\_)}$ to $\mathsf{GrayCat}$ preserves limits. $\square$

6.6. Lemma. *The face maps are natural with respect to weak maps, that is*

$$
\begin{tikzcd}
\overrightarrow{\mathbb{G}} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "\wp f"'] & \mathbb{G} \arrow[d, "f"] \\
\overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbb{H}
\end{tikzcd} \tag{156}
$$

commutes.

Proof We write (156) in terms of its underlying maps:

$$
\begin{tikzcd}
\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "d"] \arrow[d, "d"] & \mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1 e"] & \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, shift left, "\mathrm{Q}^1 d_0"] \arrow[r, shift right, "\mathrm{Q}^1 d_1"'] \arrow[d, "\psi"] & \mathrm{Q}^1\mathbb{G} \arrow[d, equal] \\
\mathrm{Q}^1\mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[r, "\mathrm{Q}^1\psi"'] \arrow[urr, "e"] & \mathrm{Q}^1\overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, "e"] \arrow[d, "\mathrm{Q}^1\overrightarrow{f}"] & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "\overrightarrow{f}"] & \mathrm{Q}^1\mathbb{G} \arrow[d, "f"] \\
& \mathrm{Q}^1\overrightarrow{\mathbb{H}} \arrow[r, "e"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbb{H}
\end{tikzcd} , \tag{157}
$$

that is, (156) commuting is equivalent to the outer frame in (157) commuting. All parts are given by naturality and the co-unit laws of Q^1, except the upper right square.

We use remark 3.10 to conclude $d_0\psi = \mathrm{Q}^1 d_0$ and $d_1\psi = \mathrm{Q}^1 d_1$: By naturality and semi-distributivity we get $ed_0\psi = d_0\overrightarrow{e}\,\psi = d_0 e = e\mathrm{Q}^1 d_0$, furthermore $(d_0\psi)_1 = (\mathrm{Q}^1 d_0)_1$ is immediate from the definition of ψ. The map d_1 is obviously treated in the same way. $\square$

6.7. Lemma. *The degeneracy maps of the path space are natural with respect to weak maps:*

$$\begin{tikzcd} \mathbb{G} \arrow[r, "i"] \arrow[d, "f"'] & \overrightarrow{\mathbb{G}} \arrow[d, "\wp f"] \\ \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \end{tikzcd} \quad . \tag{158}$$

Proof Consider

$$\begin{tikzcd} \mathrm{Q}^1\mathbb{G} \arrow[r, "d"] \arrow[d, "d"'] & \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1 e"] & \mathrm{Q}^1\mathbb{G} \arrow[r, "\mathrm{Q}^1 i"] \arrow[d, equal] & \mathrm{Q}^1\overrightarrow{\mathbb{G}} \arrow[d, "\psi"] \\ \mathrm{Q}^1\mathrm{Q}^1\mathbb{G} \arrow[rr, "e"] \arrow[d, "\mathrm{Q}^1 f"'] & & \mathrm{Q}^1\mathbb{G} \arrow[r, "i"] \arrow[d, "f"] & \overrightarrow{\mathrm{Q}^1\mathbb{G}} \arrow[d, "\overrightarrow{f}"] \\ \mathrm{Q}^1\mathbb{H} \arrow[rr, "e"'] & & \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \end{tikzcd} \quad . \tag{159}$$

We conclude that then top right square commutes by computing $\overrightarrow{e}i = ie = e\mathrm{Q}^1 i = \overrightarrow{e}\psi\mathrm{Q}^1 i$ and checking that $(\psi\mathrm{Q}^1 i)_1 = i_1$ and again applying remark 3.10 together with lemma 6.2. $\square$

The funtor $\wp$ can also be applied to Q^1-graph maps by setting $\wp' = (\wp\tilde{\mathbb{G}})^\vee$; see lemma 3.21 for the notation. For the sake of completeness we describe briefly the effect of $\wp'$ at the level of 1-cells as well as its 2-co-cycle. Let $G\colon \mathbb{G} \longrightarrow \mathbb{H}$ be a Q^1-graph map. We take a 1-cell $g\colon f \longrightarrow f'$ from $\overrightarrow{\mathbb{G}}$ and calculate:

$$(\wp' G)(g) = \left(\vec{G}\psi\right)^\vee(g) = \vec{G}\psi\left[\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "\swarrow g_2"] & \cdot \arrow[d, "g_1"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd}\right]$$

$$= \left(\begin{tikzcd} \cdot \arrow[r, "{\tilde{G}[f]}"] \arrow[d, "{\tilde{G}[g_0]}"'] \arrow[dr, phantom, "{\swarrow \tilde{G}(g_2;[g_1,f],[f,g_0])}"] & \cdot \arrow[d, "{\tilde{G}[g_1]}"] \\ \cdot \arrow[r, "{\tilde{G}[f']}"'] & \cdot \end{tikzcd}\right) = \left(\begin{tikzcd} \cdot \arrow[r, "Gf"] \arrow[d, "Gg_0"'] \arrow[dr, phantom, "{\swarrow \overline{G^2_{f',g_0}}\ \#_1 Gg_2\ \#_1 G^2_{g_1,f}}"] & \cdot \arrow[d, "Gg_1"] \\ \cdot \arrow[r, "Gf'"'] & \cdot \end{tikzcd}\right) \tag{160}$$

Taking two composable 1-cells $g\colon f \longrightarrow f'$ and $h\colon f' \longrightarrow f''$ of $\overrightarrow{\mathbb{G}}$ we get a 2-cocycle with components as shown in (161), where in the end the $\tilde{G}\kappa_{\dots}$ are iterated 2-cocycles of G.

6.8. Iterating the Path Space Construction.

$$((\wp' G)^\vee)^2_{h,g} = ((\overrightarrow{\tilde{G}}\psi)^\vee)^2_{h,g} = \overrightarrow{\tilde{G}}\psi(\kappa_{h,g})$$

$$= \overrightarrow{\tilde{G}}\psi\left(
\begin{array}{c}
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "f"] \arrow[d, bend right=80, "h_0\#_0 g_0"'] \arrow[d, "h_0\#_0 g_0"] \arrow[d, phantom, bend right=40, "\overset{\Leftarrow}{\mathrm{id}}"] \arrow[dr, phantom, "\substack{(h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2)}"] & \cdot \arrow[d, "h_1\#_0 g_1"] \\
\cdot \arrow[r, "f''"'] & \cdot
\end{tikzcd} \\
\Big\Downarrow \mathrm{id}_{(h_2\#_0 g_0)\#_1(h_1\#_0 g_2)}; \\
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "f"] \arrow[d, "h_0\#_0 g_0"'] \arrow[dr, phantom, "\substack{(h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2)}"] & \cdot \arrow[d, "h_1\#_0 g_1"'] \arrow[d, bend left=80, "h_1\#_0 g_1"] \arrow[d, phantom, bend left=40, "\overset{\Leftarrow}{\mathrm{id}}"] \\
\cdot \arrow[r, "f''"'] & \cdot
\end{tikzcd}
\end{array}
\left[
\begin{tikzcd}
\cdot \arrow[r, "f'"] \arrow[d, "h_0"'] \arrow[dr, phantom, "h_2"] & \cdot \arrow[d, "h_1"] \\
\cdot \arrow[r, "f''"'] & \cdot
\end{tikzcd}
,
\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d, "g_0"'] \arrow[dr, phantom, "g_2"] & \cdot \arrow[d, "g_1"] \\
\cdot \arrow[r, "f'"'] & \cdot
\end{tikzcd}
\right],
\left[
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "f"] \arrow[d, "h_0\#_0 g_0"'] \arrow[dr, phantom, "\substack{(h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2)}"] & \cdot \arrow[d, "h_1\#_0 g_1"] \\
\cdot \arrow[r, "f''"'] & \cdot
\end{tikzcd}
\right]
\right)$$

$$= \overrightarrow{\tilde{G}}\left(
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "{[f]}"] \arrow[d, bend right=80, "{[h_0\#_0 g_0]}"'] \arrow[d, "{[h_0,g_0]}"] \arrow[d, phantom, bend right=40, "\overset{\Leftarrow}{\kappa_{h_0,g_0}}"] \arrow[dr, phantom, "\substack{((h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2);\\ {[h_1,g_1,f]},\\ {[f'',h_0,g_0]})}"] & \cdot \arrow[d, "{[h_1,g_1]}"] \\
\cdot \arrow[r, "{[f'']}"'] & \cdot
\end{tikzcd}
\xRrightarrow{\substack{(\mathrm{id}_{(h_2\#_0 g_0)\#_1(h_1\#_0 g_2)};\\ (h_2\#_0 g_0)\#_1(h_1\#_0 g_2),\\ (h_2\#_0 g_0)\#_1(h_1\#_0 g_2);\\ {[h_1,g_1,f]},{[f'',h_0\#_0 g_0]})}}
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "{[f]}"] \arrow[d, "{[h_0\#_0 g_0]}"'] \arrow[dr, phantom, "\substack{((h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2);\\ {[h_1\#_0 g_1,f]},\\ {[f'',h_0\#_0 g_0]})}"] & \cdot \arrow[d, "{[h_1\#_0 g_1]}"'] \arrow[d, bend left=80, "{[h_1,g_1]}"] \arrow[d, phantom, bend left=40, "\overset{\Leftarrow}{\kappa_{h_1,g_1}}"] \\
\cdot \arrow[r, "{[f'']}"'] & \cdot
\end{tikzcd}
\right)$$

$$= \left(
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "{\tilde{G}[f]}"] \arrow[d, bend right=80, "{\tilde{G}[h_0\#_0 g_0]}"'] \arrow[d, "{\tilde{G}[h_0,g_0]}"] \arrow[d, phantom, bend right=40, "\overset{\Leftarrow}{\tilde{G}\kappa_{h_0,g_0}}"] \arrow[dr, phantom, "\substack{\tilde{G}((h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2);\\ {[h_1,g_1,f]},\\ {[f'',h_0,g_0]})}"] & \cdot \arrow[d, "{\tilde{G}[h_1,g_1]}"] \\
\cdot \arrow[r, "{\tilde{G}[f'']}"'] & \cdot
\end{tikzcd}
\xRrightarrow{\substack{\tilde{G}(\mathrm{id}_{(h_2\#_0 g_0)\#_1(h_1\#_0 g_2)};\\ (h_2\#_0 g_0)\#_1(h_1\#_0 g_2),\\ (h_2\#_0 g_0)\#_1(h_1\#_0 g_2);\\ {[h_1,g_1,f]},{[f'',h_0\#_0 g_0]})}}
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "{\tilde{G}[f]}"] \arrow[d, "{\tilde{G}[h_0\#_0 g_0]}"'] \arrow[dr, phantom, "\substack{\tilde{G}((h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2);\\ {[h_1\#_0 g_1,f]},\\ {[f'',h_0\#_0 g_0]})}"] & \cdot \arrow[d, "{\tilde{G}[h_1\#_0 g_1]}"'] \arrow[d, bend left=80, "{\tilde{G}[h_1,g_1]}"] \arrow[d, phantom, bend left=40, "\overset{\Leftarrow}{G\kappa_{h_1,g_1}}"] \\
\cdot \arrow[r, "{\tilde{G}[f'']}"'] & \cdot
\end{tikzcd}
\right)$$

$$= \left(
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "Gf"] \arrow[d, bend right=80, "G(h_0\#_0 g_0)"'] \arrow[d, "Gh_0\#_0 Gg_0"] \arrow[d, phantom, bend right=40, "\overset{\Leftarrow}{G^2_{h_0,g_0}}"] \arrow[dr, phantom, "\substack{\overline{\tilde{G}\kappa_{f'',h_0,g_0}}\\ \#_1 G((h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2))\\ \#_1\tilde{G}\kappa_{h_1,g_1,f}}"] & \cdot \arrow[d, "Gh_1\#_0 Gg_1"] \\
\cdot \arrow[r, "Gf''"'] & \cdot
\end{tikzcd}
\xRrightarrow{\mathrm{id}}
\begin{tikzcd}[column sep=huge, row sep=huge]
\cdot \arrow[r, "Gf"] \arrow[d, "G(h_0\#_0 g_0)"'] \arrow[dr, phantom, "\substack{G((h_2\#_0 g_0)\\ \#_1(h_1\#_0 g_2);\\ {[h_1\#_0 g_1,f]},\\ {[f'',h_0\#_0 g_0]})}"] & \cdot \arrow[d, "G(h_1\#_0 g_1)"'] \arrow[d, bend left=80, "Gh_1\#_0 Gg_1"] \arrow[d, phantom, bend left=40, "\overset{\Leftarrow}{G\kappa_{h_1,g_1}}"] \\
\cdot \arrow[r, "{\tilde{G}[f'']}"'] & \cdot
\end{tikzcd}
\right) \tag{161}$$

6.9. Lemma. *The maps i, d_0, d_1 and m for all* Gray*-categories $\mathbb{H}$ constitute natural transformations with respect to strict maps.* $\square$

For reference, this means that for all $f \colon \mathbb{H} \longrightarrow \mathbb{K}$ the following diagram commutes sequentially:

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{\mathbb{H}} \overrightarrow{\mathbb{H}} \arrow[r, "m"] \arrow[d, "\overrightarrow{f} \times \overrightarrow{f}"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, "\overrightarrow{f}"'] & \mathbb{H} \arrow[l, "i" description] \arrow[d, "f"] \\
\overrightarrow{\mathbb{K}} \times_{\mathbb{K}} \overrightarrow{\mathbb{K}} \arrow[r, "m"'] & \overrightarrow{\mathbb{K}} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{K} \arrow[l, "i" description]
\end{tikzcd} \tag{162}$$

Iterating the arrow construction yields an internal cubical set, so it allows us to talk about higher cells in the internal language of GrayCat. But since we want to construct an internal Gray-category we need to restrict to cubical cells with certain degeneracies. E.g. the 0-cells in $\overrightarrow{\overrightarrow{\mathbb{H}}}$ are squares, and we want to filter out those square that are actually bigons, that is, have identity arrow as left and right sides. That is exactly what we get by forming the double pullback on the left:

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \arrow[r, "j"] \arrow[d, shift right=2, "\overline{d_0}"'] \arrow[d, shift left=2, "\overline{d_1}"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, shift right=2, "\overrightarrow{d_0}"'] \arrow[d, shift left=2, "\overrightarrow{d_1}"] & \overrightarrow{\mathbb{H}} \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
\mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H}
\end{tikzcd} \tag{163}$$

where $\overline{\overline{\mathbb{H}}}$ is the intersection of the pullbacks of d_0 and d_1 along i. Let $d_{0,1}^j = d_{0,1} j$.

6.10. Lemma. *The diagram*

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \arrow[r, shift left=2, "d_1^j"] \arrow[r, shift right=2, "d_0^j"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H}
\end{tikzcd} \tag{164}$$

is a globular object, i.e. $d_0 d_0^j = d_0 d_1^j$ and $d_1 d_0^j = d_1 d_1^j$.

Proof Using the naturality of d_0 and d_1 we calculate:

$$d_0 d_0^j = d_0 d_0 j = d_0 \overrightarrow{d_0} j = d_0 i \overline{d_0} = d_1 i \overline{d_0} = d_1 \overrightarrow{d_0} j = d_0 d_1 j = d_1 d_0^j \,, \tag{165}$$

and similarly for d_1. $\square$

To get a unit for $\overline{\overline{\mathbb{H}}}$ we consider the following diagram:

$$
\begin{tikzcd}
\overrightarrow{\mathbb{H}} \arrow[drr, "i"] \arrow[dr, dotted, "\bar{i}"] \arrow[ddr, shift right, "d_0"'] \arrow[ddr, shift left, "d_1"] & & & \\
& \overline{\overline{\mathbb{H}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d_0}"'] \arrow[d, shift left, "\overline{d_1}"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\overrightarrow{d_0}"'] \arrow[d, shift left, "\overrightarrow{d_1}"] & \overrightarrow{\mathbb{H}} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
& \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}
\end{tikzcd}
\tag{166}
$$

the upper left span is a compatible source by the naturality of i. The induced arrow $\bar{i}$ is a joint section of d_0^j and d_1^j. Hence we get:

6.11. LEMMA. *The diagram*

$$
\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \arrow[r, shift left=2, "d_1^j"] \arrow[r, shift right=2, "d_0^j"'] & \overrightarrow{\mathbb{H}} \arrow[l, "\bar{i}"']
\end{tikzcd}
\tag{167}
$$

is a reflexive graph. □

6.12. LEMMA. *The mapping* $\overline{\overline{(-)}}$ *extends to a sub-functor of* $\overrightarrow{\overrightarrow{(-)}}$: $\mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}$ *with natural embedding* j.

PROOF For each $\mathbb{H}$ the map j is a monomorphism by construction and $\overline{\overline{(-)}}$ extends to morphisms by the universal property. □

6.13. LEMMA. *There is a multiplication*

$$
\overline{\overline{\mathbb{H}}} \times_{d_0^j, d_1^j} \overline{\overline{\mathbb{H}}} \xrightarrow{\overline{m}} \overline{\overline{\mathbb{H}}}
\tag{168}
$$

uniquely induced by $m_{\overrightarrow{\mathbb{H}}}$.

PROOF All we need to show is that $m(j \times j)$ factors through j, that is, show that the two outer rectangles commute:

$$
\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{d_0^j, d_1^j} \overline{\overline{\mathbb{H}}} \arrow[r, "j \times j"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \arrow[d, dotted, bend left=60, "\overline{m}"] \arrow[dd, bend right=60, shift right, "d_0'"'] \arrow[dd, bend right=60, shift left, "d_1'"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \times_{d_0, d_1} \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[d, "m"] \\
\overline{\overline{\mathbb{H}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d}_0"'] \arrow[d, shift left, "\overline{d}_1"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[d, shift right, "\overrightarrow{d}_0"'] \arrow[d, shift left, "\overrightarrow{d}_1"] \\
\mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd}
\tag{169}
$$

First we prove that $\overline{d}_0 p_0 = \overline{d}_0 p_1$:

$$\overline{d}_0 p_0 = d_0 i \overline{d}_0 p_0 = d_0 \overrightarrow{d}_0 j p_0 = d_0 d_0 j p_0 = d_0 d_0^j p_0 = d_0 d_1^j p_1 = d_0 d_0^j p_1 = \overline{d}_0 p_1 \tag{170}$$

which holds by (167), (164) and (163). Similarly $\overline{d}_1 p_0 = \overline{d}_1 p_1$. Thus we may define $d_0' = \overline{d}_0 p_0$ and $d_1' = \overline{d}_1 p_0$. Note that $j \times j$ is universally induced by $d_0 j p_0 = d_1 j p_1$.

Furthermore we need that $(i\overline{d}_0 \times i\overline{d}_0) = (i,i)d_0'$ and $(i\overline{d}_1 \times i\overline{d}_1) = (i,i)d_1'$. Consider

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{d_0^j, d_1^j} \overline{\overline{\mathbb{H}}} \arrow[rrr, "p_1"] \arrow[dr, "d_0'"] \arrow[ddd, "p_0"'] \arrow[ddrr, bend right, "(i\overline{d}_0 \times i\overline{d}_0)"'] & & & \overline{\overline{\mathbb{H}}} \arrow[d, "\overline{d}_0"] \\
& \mathbb{H} \arrow[rr] \arrow[dr, "{(i,i)}"] \arrow[dd] & & \mathbb{H} \arrow[d, "i"] \\
& & \overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r] \arrow[d] & \overrightarrow{\mathbb{H}} \arrow[d, "d_1"] \\
\overline{\overline{\mathbb{H}}} \arrow[r, "\overline{d}_0"'] & \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \arrow[r, "d_0"'] & \mathbb{H}
\end{tikzcd} \tag{171}$$

The top and left squares commute by (170) and (164) makes the pair $(i\overline{d}_0 p_0, i\overline{d}_0 p_1)$ a Compatible source for lower right pullback square. The universality thus proves our equation.

Finally we verify that

$$\overrightarrow{d}_0 m(j \times j) = m(\overrightarrow{d}_0 \times \overrightarrow{d}_0)(j \times j) = m(\overrightarrow{d}_0 j \times \overrightarrow{d}_0 j) = m(i\overline{d}_0 j \times i\overline{d}_0 j) = m(i,i)d_0' = id_0' \tag{172}$$

By the same token $d_1 m(j \times j) = id_1'$ hence we get the desired $\overline{m}$. □

6.14. Lemma. *The composition $\overline{m}$ is unital and associative, that is, it makes* (167) *a category.*

Proof Obvious since $m_{\overrightarrow{\mathbb{H}}}$ is so. □

6.15. Remark. *By lemma 4.19 the applying $\overrightarrow{(\_)}$ to an internal category*

$$\begin{tikzcd}
\mathbb{K} \times_{d_0,d_1} \mathbb{K} \arrow[r, "m"] & \mathbb{K} \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H} \arrow[l, "i" description]
\end{tikzcd} \tag{173}$$

yields an internal category

$$\begin{tikzcd}
\overrightarrow{\mathbb{K} \times_{d_0,d_1} \mathbb{K}} \arrow[r, "\overrightarrow{m}"] & \overrightarrow{\mathbb{K}} \arrow[r, shift left=2, "\overrightarrow{d_1}"] \arrow[r, shift right=2, "\overrightarrow{d_0}"'] & \overrightarrow{\mathbb{H}} \arrow[l, "\overrightarrow{i}" description]
\end{tikzcd} . \tag{174}$$

6.16. Lemma. *There is a right whiskering map*

$$\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \xrightarrow[\;/\;]{w_r} \overline{\overline{\mathbb{H}}} \tag{175}$$

induced uniquely by $\overrightarrow{m}$.

By the same argument there is a corresponding left whiskering map w_ℓ.

Proof We construct a restricted horizontal composition $m'_r \colon \overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \nrightarrow \overrightarrow{\overrightarrow{\mathbb{H}}}$ in the following diagram:

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\bar{d}_1} \overline{\overline{\mathbb{H}}} \arrow[r, "p_1"] \arrow[d, "p_0"] \arrow[drr, dashed, "i\times j"] \arrow[dddrrrr, bend right, "/" marking, "m'_r"'] & \overrightarrow{\mathbb{H}} \arrow[d, "d_0"] \arrow[drr, "i"] & & & \\
\overline{\overline{\mathbb{H}}} \arrow[r, "\bar{d}_1"'] \arrow[drr, "j"'] & \mathbb{H} \arrow[drr, "i"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \times_{\vec{d}_0,\vec{d}_1} \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r] \arrow[d] \arrow[ddrr, "\vec{m}"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, "\vec{d}_1"] \arrow[d, "\vec{d}_0"] & \overrightarrow{\mathbb{H}} \\
 & & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, "\vec{d}_1"'] \arrow[d, "\vec{d}_0"'] & \overrightarrow{\overrightarrow{\mathbb{H}}} & \\
 & & \overrightarrow{\mathbb{H}} & & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[ll, "\vec{d}_0"] \arrow[uu, "\vec{d}_1"']
\end{tikzcd} \tag{176}$$

where $i \times j$ is universally induced and m'_r is defined as the composite $\overrightarrow{m}(i \times j)$. We need to show that m'_r factors through $\overline{\overline{\mathbb{H}}}$.

Consider the defining pullback for $\overline{\overline{\mathbb{H}}}$:

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[dr, dotted, "/" marking, "w_r"] \arrow[drr, "/" marking, "m'_r"] \arrow[ddr, shift right, "\overline{d_0}p_0"'] \arrow[ddr, shift left, "d_1 p_1"] & & & \\
 & \overline{\overline{\mathbb{H}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d_0}"'] \arrow[d, shift left, "\overline{d_1}"] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\vec{d_0}"'] \arrow[d, shift left, "\vec{d_1}"] & \overrightarrow{\mathbb{H}} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
 & \mathbb{H} \arrow[r, "i"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}\,.
\end{tikzcd} \tag{177}$$

We need to show that $\overrightarrow{d}_0 m'_r = i\bar{d}_0 p_0$ and $\overrightarrow{d}_1 m'_r = i d_1 p_1$ to obtain a universal w_r, hence we calculate:

$$\overrightarrow{d}_0 m'_r = \overrightarrow{d}_0 \overrightarrow{m}(i \times j) = \overrightarrow{d}_0 j p_0 = \bar{d}_0 p_0 \tag{178}$$

$$\overrightarrow{d}_1 m'_r = \overrightarrow{d}_1 \overrightarrow{m}(i \times j) = \overrightarrow{d}_1 i p_1 = \bar{d}_1 p_1 \tag{179}$$

using the definitions of $i \times j$ and j as well as the naturality of i. $\square$

6.17. Lemma. *Left and right whiskering are compatible, that is, the diagram*

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\bar{d}_1} \overline{\overline{\mathbb{H}}} \times_{\bar{d}_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_r \times \overrightarrow{\mathbb{H}}"] \arrow[d, "\overrightarrow{\mathbb{H}} \times w_\ell"'] & \overline{\overline{\mathbb{H}}} \times_{\bar{d}_0,d_1} \overrightarrow{\mathbb{H}} \arrow[d, "w_\ell"] \\
\overrightarrow{\mathbb{H}} \times_{d_0,\bar{d}_1} \overline{\overline{\mathbb{H}}} \arrow[r, "w_r"'] & \overline{\overline{\mathbb{H}}}
\end{tikzcd} \tag{180}$$

commutes.

Proof The objects in the above diagram embed into $\overrightarrow{\overrightarrow{\mathbb{H}}}$ and the horizontal composition $\overrightarrow{m}$ is associative since $m_{\mathbb{H}}$ is so. $\square$

We apply the construction in (163) to (167) as follows:

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "j"] \arrow[d, shift right, "\overline{d_0^j}"'] \arrow[d, shift left, "\overline{d_1^j}"] & \overrightarrow{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "\overrightarrow{d_0^j}"'] \arrow[d, shift left, "\overrightarrow{d_1^j}"] & \overline{\overline{\mathbb{H}}} \arrow[d, shift right, "d_0^j"'] \arrow[d, shift left, "d_1^j"] \\
\overrightarrow{\mathbb{H}} \arrow[r, "i"'] & \overrightarrow{\overrightarrow{\mathbb{H}}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd} \tag{181}$$

By (167) we get a reflexive graph

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left=2, "d_1^j"] \arrow[r, shift right=2, "d_0^j"'] & \overline{\overline{\mathbb{H}}} \arrow[l, "\bar{i}" description]
\end{tikzcd} \tag{182}$$

where by (164)

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left, "d_1^j"] \arrow[r, shift right, "d_0^j"'] & \overline{\overline{\mathbb{H}}} \arrow[r, shift left, "d_1^j"] \arrow[r, shift right, "d_0^j"'] & \overrightarrow{\mathbb{H}} \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & \mathbb{H}
\end{tikzcd} \tag{183}$$

is a 3-globular object. Furthermore, by lemma 6.13 we get a vertical multiplication map

$$\overline{\overline{\overline{\mathbb{H}}}} \times_{d_0^j,d_1^j} \overline{\overline{\overline{\mathbb{H}}}} \xrightarrow{\overline{\overline{m}}} \overline{\overline{\overline{\mathbb{H}}}} \tag{184}$$

arising as a restriction of $m_{\overline{\overline{\mathbb{H}}}}$.

6.18. Lemma. *w_ℓ and w_r extend m. That is*

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "w_r"] \arrow[d, shift right, "\overrightarrow{\mathbb{H}} \times d_0^j"'] \arrow[d, shift left, "\overrightarrow{\mathbb{H}} \times d_1^j"] & \overline{\overline{\mathbb{H}}} \arrow[d, shift right, "d_0^j"'] \arrow[d, shift left, "d_1^j"] \\
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "m"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_\ell"] \arrow[d, shift right, "d_0 \times \overrightarrow{\mathbb{H}}"'] \arrow[d, shift left, "d_1 \times \overrightarrow{\mathbb{H}}"] & \overline{\overline{\mathbb{H}}} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
\overrightarrow{\mathbb{H}} \times_{d_0,d_1} \overrightarrow{\mathbb{H}} \arrow[r, "m"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd} \tag{185}$$

commute serially.

Proof Considering the proof of lemma 6.16 we calculate:

$$d_0^j w_r = d_0 j w_r = d_0 m_r' = d_0 \overrightarrow{m}(i \times j) = m(d_0 \times d_0)(i \times j) = m(\overrightarrow{\mathbb{H}} \times d_0^j)\,. \tag{186}$$

Similarly for d_1^j and w_ℓ. $\square$

Lemme 6.18 allows us to define left and right horizontal composites. Call the composite along the middle in the following diagram $h_\ell \colon \overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \nrightarrow \overline{\overline{\mathbb{H}}}$:

$$\begin{tikzcd}
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "w_r", "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} & \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[u, "d_0^j \times \overline{\overline{\mathbb{H}}}"] \arrow[d, "\overline{\overline{\mathbb{H}}} \times d_1^j"'] \arrow[r, dotted, "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} \arrow[u] \arrow[d] \arrow[r, "\overline{m}", "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_\ell"', "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} &
\end{tikzcd} , \tag{187}$$

and correspondingly $h_r \colon \overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \nrightarrow \overline{\overline{\mathbb{H}}}$:

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},d_1} \overrightarrow{\mathbb{H}} \arrow[r, "w_\ell", "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} & \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[u, "\overline{\overline{\mathbb{H}}} \times d_1^j"] \arrow[d, "d_0^j \times \overline{\overline{\mathbb{H}}}"'] \arrow[r, dotted, "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} \times_{d_0^j,d_1^j} \overline{\overline{\mathbb{H}}} \arrow[u] \arrow[d] \arrow[r, "\overline{m}", "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} \\
\overrightarrow{\mathbb{H}} \times_{d_0,\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "w_r"', "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} &
\end{tikzcd} . \tag{188}$$

6.19. Lemma. *Left and right horizontal composites give a globular object*

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, shift left, "h_\ell", "\shortmid"{marking}] \arrow[r, shift right, "h_r"', "\shortmid"{marking}] & \overline{\overline{\mathbb{H}}} \arrow[r, shift left, "d_1^j"] \arrow[r, shift right, "d_0^j"'] & \overrightarrow{\mathbb{H}}
\end{tikzcd} . \tag{189}$$

Proof We calculate:

$$
\begin{aligned}
d_0^j h_\ell &\overset{(187)}{=} d_0 j \overline{m} \left\langle w_r(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}), w_\ell(\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j) \right\rangle && (190)\\
&\overset{(169)}{=} d_0 m(j \times j) \left\langle w_r(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}), w_\ell(\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j) \right\rangle && (191)\\
&\overset{(177)}{=} d_0 p_0 \left\langle m_r'(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}), m_\ell'(\overline{\overline{\overline{\mathbb{H}}}} \times d_1^j) \right\rangle && (192)\\
&= d_0 m_r'(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}) && (193)\\
&= d_0 \overrightarrow{m}(i \times j)(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}) && (194)\\
&\overset{(156)}{=} m(d_0 \times d_0)(i \times j)(d_0^j \times \overline{\overline{\overline{\mathbb{H}}}}) && (195)\\
&= m(d_0^j \times d_0^j) && (196)
\end{aligned}
$$

and by the same token

$$d_0^j h_r = m(d_0^j \times d_0^j)\,. \qquad (197)$$

Analogously for d_1^j. □

6.20. The Space of Parallel Cells. For a Gray-category $\mathbb{H}$ we define the space of parallel 1-cells $P^1(\mathbb{H})$ as the following limit:

$$
\begin{tikzcd}
& P^1(\mathbb{H}) \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
\overrightarrow{\mathbb{H}} \arrow[d, "d_0"'] \arrow[drr, "d_1"' near start] & & \overrightarrow{\mathbb{H}} \arrow[d, "d_1"] \arrow[dll, "d_0" near start] \\
\mathbb{H} & & \mathbb{H}
\end{tikzcd}
\qquad (198)
$$

$$
\begin{tikzcd}
& P^2(\mathbb{H}) \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
\overline{\overline{\mathbb{H}}} \arrow[d, "d_0^j"'] \arrow[drr, "d_1^j"' near start] & & \overline{\overline{\mathbb{H}}} \arrow[d, "d_1^j"] \arrow[dll, "d_0^j" near start] \\
\overrightarrow{\mathbb{H}} & & \overrightarrow{\mathbb{H}}
\end{tikzcd}
\qquad (199)
$$

6.21. Lemma. *The canonical map* $\left\langle d_0^j, d_1^j \right\rangle : \overline{\overline{\overline{\mathbb{H}}}} \longrightarrow P^2(\mathbb{H})$ *is 1-Cartesian.*

Proof Consider the following cells in $\overline{\overline{\overline{\mathbb{H}}}}$

$$f = (f_4; f_2, f_3; f_0, f_1) \tag{200}$$
$$g = (g_4; g_2, g_3; g_0, g_1) \tag{201}$$
$$h = (h_4, h_5; h_2, h_3; h_0, h_1) \colon f \longrightarrow g \tag{202}$$
$$k = (k_4, k_5; k_2, k_3; k_0, k_1) \colon f \longrightarrow g \tag{203}$$
$$\alpha = (\alpha_3; \alpha_1, \alpha_2) \colon h \Longrightarrow k \tag{204}$$

By construction the map $\left\langle d_0^j, d_1^j \right\rangle$ acts on this data as follows:

$$f \mapsto ((f_2; f_0, f_1), (f_3; f_0, f_1)) \tag{205}$$
$$g \mapsto ((g_2; g_0, g_1), (g_3; g_0, g_1)) \tag{206}$$
$$h \mapsto ((h_4; h_2, h_3; h_0, h_1), (h_5; h_2, h_3; h_0, h_1)) \tag{207}$$
$$k \mapsto ((k_4; k_2, k_3; k_0, k_1), (k_5; k_2, k_3; k_0, k_1)) \tag{208}$$
$$\alpha \mapsto ((\alpha_3; \alpha_1, \alpha_2), (\alpha_3; \alpha_1, \alpha_2)) \tag{209}$$

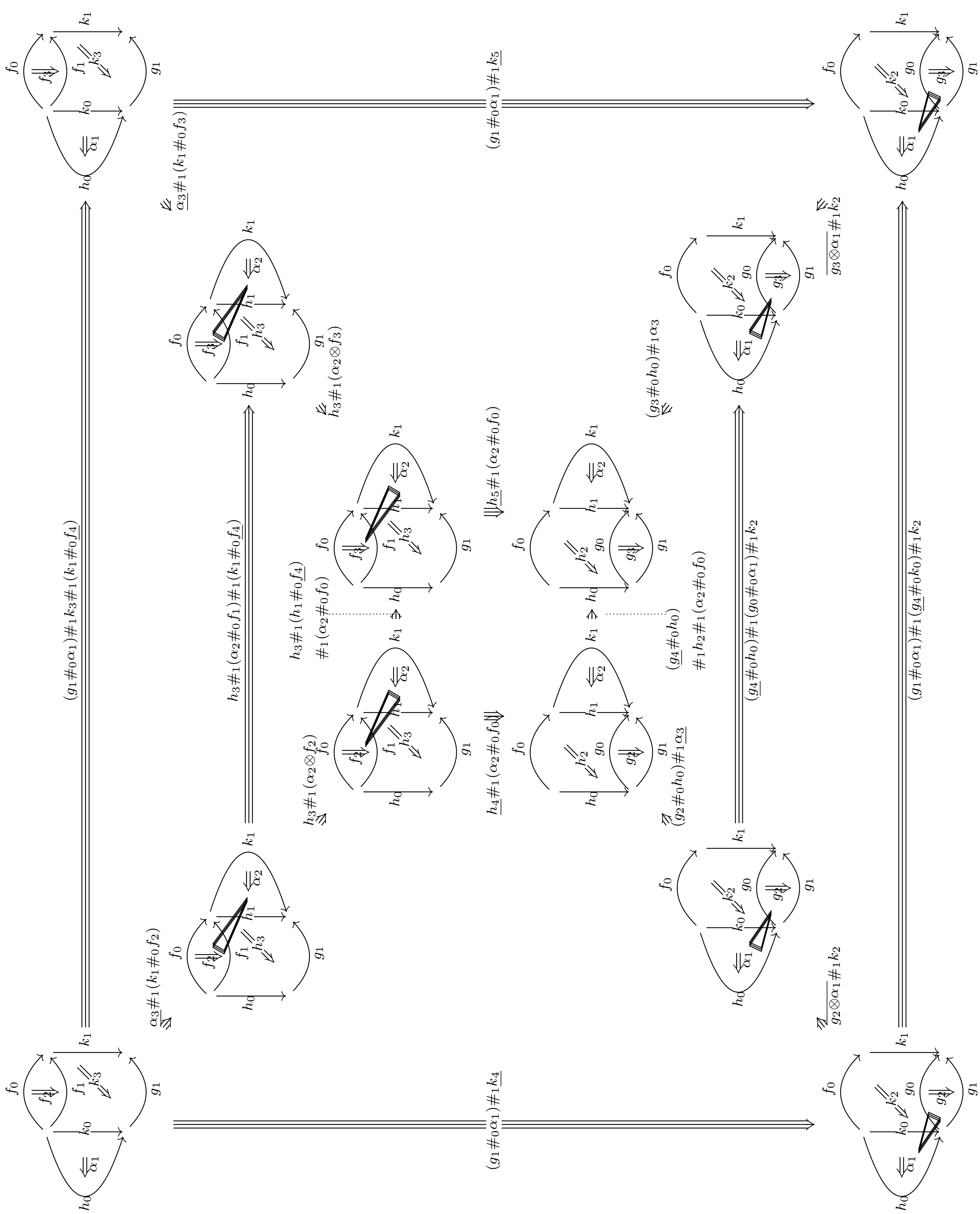
where on the right we find parallel pairs of cells from $\overline{\overline{\mathbb{H}}}$, that is, in (210) the central square, the outer square, and the left and right trapezoids commute by assumption.

The requisite compatibility conditions for f, g, h, k, α to be cells of $\overline{\overline{\overline{\mathbb{H}}}}$ are displayed in (210). We obverse that the remaining trapezoids at the top and the bottom commute by naturality of $\#_1$ and $\otimes$ in $\mathbb{H}$. Hence we conclude that given 1-cells h, k in $\overline{\overline{\overline{\mathbb{H}}}}$ all higher cells, including 3-cells, between them are determined by their image under $\left\langle d_0^j, d_1^j \right\rangle$. $\square$

6.22. The Tensor Map. Given that by lemma 6.21 we have a 1-Cartesian map $\left\langle d_0^j, d_1^j \right\rangle \overline{\overline{\overline{\mathbb{H}}}} \longrightarrow P^2(\mathbb{H})$ we consider the following diagram in $\mathsf{GrayCat}_{\mathrm{Q}^1}$

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\overline{\mathbb{H}}}} \arrow[dr, dotted, "t"'] \arrow[drr, "\langle h_\ell, h_r \rangle"] & & \\
& \overline{\overline{\overline{\mathbb{H}}}} \arrow[r, "\left\langle d_0^j, d_1^j \right\rangle"'] & P^2(\mathbb{H})
\end{tikzcd} \tag{211}$$

where h_ℓ and h_r are given by (187) and (188) respectively. By (189) we know that (h_ℓ, h_r) is a source for (199) hence we obtain $\langle h_\ell, h_r \rangle$.


$(g_1\#_0\alpha_1)\#_1\underline{k_5}$
$(g_1\#_0\alpha_1)\#_1\underline{k_4}$
$(g_1\#_0\alpha_1)\#_1k_3\#_1(k_1\#_0\underline{f_4})$
$(g_1\#_0\alpha_1)\#_1(\underline{g_4}\#_0k_0)\#_1k_2$
$\underline{\alpha_3}\#_1(k_1\#_0f_3)$
$\underline{\alpha_3}\#_1(k_1\#_0f_2)$
$h_3\#_1(\alpha_2\#_0f_1)\#_1(k_1\#_0\underline{f_4})$
$h_3\#_1(\alpha_2\otimes f_3)$
$\underline{h_3}\#_1(\alpha_2\otimes\underline{f_2})$
$h_3\#_1(h_1\#_0\underline{f_4})$
$\#_1(\alpha_2\#_0f_0)$
$\underline{h_5}\#_1(\alpha_2\#_0f_0)$
$\underline{h_4}\#_1(\alpha_2\#_0f_0)$
$(\underline{g_4}\#_0h_0)$
$\#_1h_2\#_1(\alpha_2\#_0f_0)$
$(\underline{g_3}\#_0h_0)\#_1\alpha_3$
$(g_2\#_0h_0)\#_1\underline{\alpha_3}$
$(\underline{g_4}\#_0h_0)\#_1(g_0\#_0\alpha_1)\#_1k_2$
$\overline{g_3\otimes\alpha_1}\#_1k_2$
$\overline{g_2\otimes\alpha_1}\#_1k_2$

(210)

There is a map $t_1\colon (\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}})_1 \longrightarrow (\overline{\overline{\overline{\mathbb{H}}}})_1$ in $\mathsf{Cat}_{\mathbb{Q}^1}$ given by:

$$(g,f) = ((g_2;g_0,g_1),(f_2;f_0,f_1)) = \left(\begin{tikzcd} {} \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow f_2"] & {} \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow g_2"] & {} \end{tikzcd}\right)$$

$$\mapsto (g_2\otimes f_2; g_2 \lhd f_2, g_2 \rhd f_2; g_0\#_0 f_0, g_1\#_0 f_1) = \left(\begin{tikzcd} {} \arrow[r, bend left, "g_0\#_0f_0"] \arrow[r, bend right, "g_1\#_0f_1"'] \arrow[r, phantom, "\Downarrow g_2\lhd f_2"] & {} \arrow[r, phantom, "\overset{g_2\otimes f_2}{\Rrightarrow}"] & {} \arrow[r, bend left, "g_0\#_0f_0"] \arrow[r, bend right, "g_1\#_0f_1"'] \arrow[r, phantom, "\Downarrow g_2\rhd f_2"] & {} \end{tikzcd}\right) \tag{212}$$

and

$$\big((k,h)\colon (g,f) \longrightarrow (g',f')\big) = \begin{pmatrix}(k_4;k_2,k_3;h_1,k_1),\\(h_4;h_2,h_3;h_0,h_1)\end{pmatrix} = \left(\begin{array}{c} \begin{tikzcd} {} \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow f_2"] \arrow[d, "h_0"'] & {} \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow g_2"] \arrow[d, "h_1"] \arrow[dl, phantom, "\swarrow h_3"] & {} \arrow[d, "k_1"] \arrow[dl, phantom, "\swarrow k_3"] \\ {} \arrow[r, bend right, "f_1'"'] & {} \arrow[r, bend right, "g_1'"'] & {} \end{tikzcd} \\ \Big\Downarrow (k_4,h_4) \\ \begin{tikzcd} {} \arrow[r, bend left, "f_0"] \arrow[d, "h_0"'] & {} \arrow[r, bend left, "g_0"] \arrow[d, "h_1"] \arrow[dl, phantom, "\swarrow h_2"] & {} \arrow[d, "k_1"] \arrow[dl, phantom, "\swarrow k_2"] \\ {} \arrow[r, bend left, "f_0'"] \arrow[r, bend right, "f_1'"'] \arrow[r, phantom, "\Downarrow f_2'"] & {} \arrow[r, bend left, "g_0'"] \arrow[r, bend right, "g_1'"'] \arrow[r, phantom, "\Downarrow g_2'"] & {} \end{tikzcd} \end{array}\right)$$

$$\mapsto \begin{pmatrix}\omega_1,\omega_2; (g_0'\#_0 h_2)\#_1(k_2\#_0 f_0),\\ (g_1'\#_0 h_3)\#_1(k_3\#_0 f_1); h_0, k_1\end{pmatrix}, \tag{213}$$

where ω_1 and ω_2 are defined as the vertical composites in (214), by definition these constitute the components of a 1-cell in $\overline{\overline{\overline{\mathbb{H}}}}$.

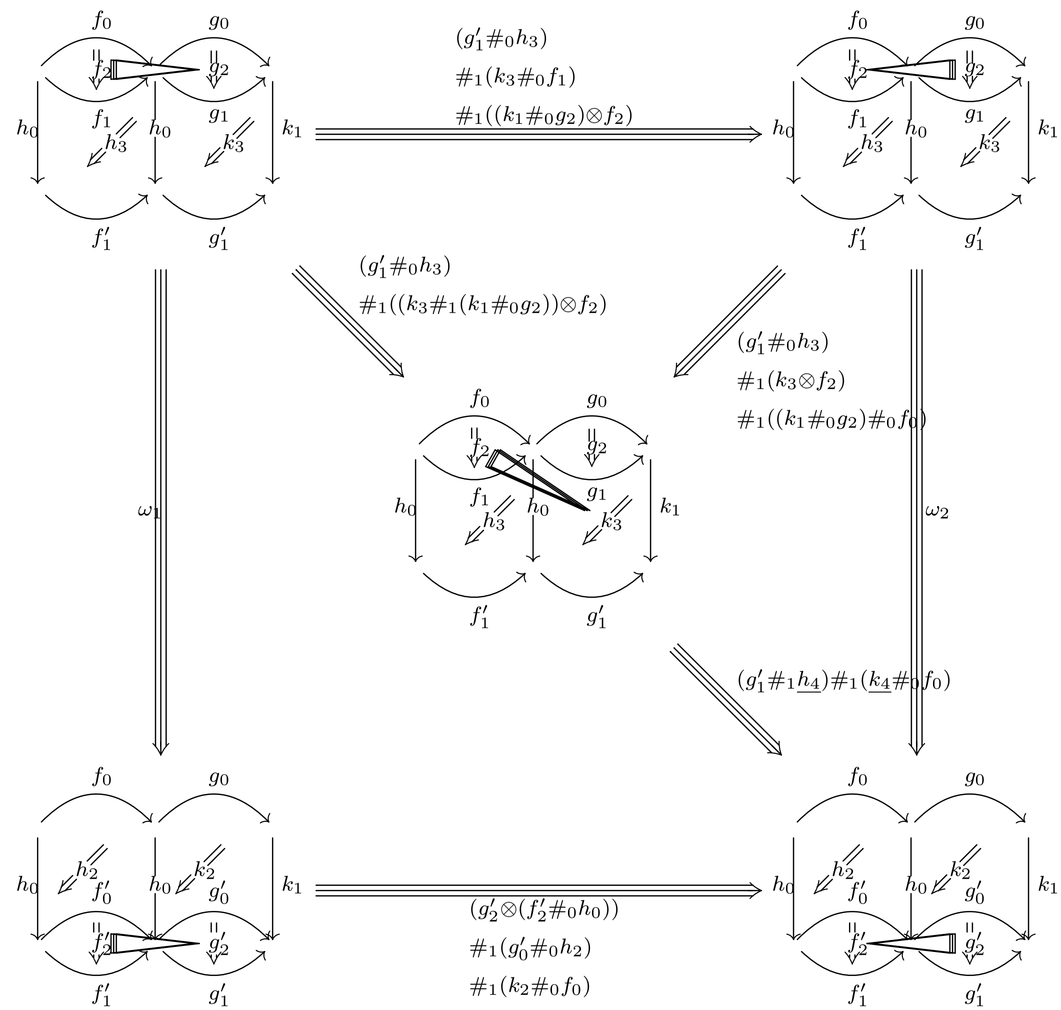
such that

6.23. LEMMA. *$\langle h_\ell, h_r\rangle_1 = \langle d_0^j, d_1^j\rangle_1 t_1$ in* RGrph.

PROOF One checks that $(h_\ell)_1 = (d_0^j t)_1$ and $(h_r)_1 = (d_1^j t)_1$ as graph maps using definitions (187) and (188). □

6.24. LEMMA. *The 3-globular set*

$$P^2(\mathbb{H}) \underset{p_0}{\overset{p^1}{\underset{\longleftarrow\Delta}{\rightrightarrows}}} \overline{\overline{\mathbb{H}}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i}{\rightrightarrows}}} \overrightarrow{\mathbb{H}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i}{\rightrightarrows}}} \mathbb{H} \tag{215}$$



$(g_1'\#_0h_3)$
$\#_1(k_3\#_0f_1)$
$\#_1((k_1\#_0g_2)\otimes f_2)$
$(g_1'\#_0h_3)$
$\#_1((k_3\#_1(k_1\#_0g_2))\otimes f_2)$
$(g_1'\#_0h_3)$
$\#_1(k_3\otimes f_2)$
$\#_1((k_1\#_0g_2)\#_0f_0)$
ω_1
ω_2
$(g_1'\#_1\underline{h_4})\#_1(\underline{k_4}\#_0f_0)$
$(g_2'\otimes(f_2'\#_0h_0))$
$\#_1(g_0'\#_0h_2)$
$\#_1(k_2\#_0f_0)$

(214)

is an internal Gray*-cateogory.*

PROOF We already know that its three lower stages constitute a sesqui-catgory. The three top parts are trivially a 2-category. The tensor map is given by

$$\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \xrightarrow{\langle h_\ell, h_r \rangle} P^2(\mathbb{H}) \tag{216}$$

which satisfies the tensor axioms by construction. ☐

We can finally prove our desired theorem:

6.25. THEOREM. *Given a* Gray*-category* $\mathbb{H}$ *there is an internal* Gray*-category in* $\mathsf{GrayCat}_{\mathrm{Q}^1}$

$$\overline{\overline{\overline{\mathbb{H}}}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i}{\rightrightarrows}}} \overline{\overline{\mathbb{H}}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i}{\rightrightarrows}}} \overrightarrow{\mathbb{H}} \underset{d_0}{\overset{d^1}{\underset{\longleftarrow i}{\rightrightarrows}}} \mathbb{H} \tag{217}$$

with composition operations $m, \overline{m}, w_\ell, w_r, \overline{\overline{m}}$ *and tensor* t.

PROOF We have a globular map

$$\begin{tikzcd}
\overline{\overline{\overline{\mathbb{H}}}} \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, "\langle d_0^j, d_1^j \rangle"'] & \overline{\overline{\mathbb{H}}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] \arrow[d] & \overrightarrow{\mathbb{H}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] \arrow[d] & \mathbb{H} \arrow[l, "i" description] \arrow[d] \\
P^2(\mathbb{H}) \arrow[r, shift left=2, "p^1"] \arrow[r, shift right=2, "p_0"'] & \overline{\overline{\mathbb{H}}} \arrow[l, "\Delta" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \overrightarrow{\mathbb{H}} \arrow[l, "i" description] \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d_0"'] & \mathbb{H} \arrow[l, "i" description]
\end{tikzcd} \tag{218}$$

This globular map is an internal sesqui-functor in the lower and at the upper degrees, and by (211) it preverses the tensor:

$$\begin{tikzcd}
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "t"] \arrow[d] & \overline{\overline{\overline{\mathbb{H}}}} \arrow[d, "\langle d_0^j, d_1^j \rangle"] \\
\overline{\overline{\mathbb{H}}} \times_{\overline{d_0},\overline{d_1}} \overline{\overline{\mathbb{H}}} \arrow[r, "\langle h_\ell, h_r \rangle"] & P^2(\mathbb{H})
\end{tikzcd} \tag{219}$$

This proves that (217) is an internal Gray-category. ☐

7. The Internal Hom Functor

We finally define the internal hom of $\mathsf{GrayCat}_{\mathrm{Q}^1}$

$$[\mathbb{G}, \mathbb{H}] = \left(\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overline{\overline{\overline{\mathbb{H}}}}) \underset{d_{0*}}{\overset{d_{1*}}{\underset{\longleftarrow i_* }{\rightrightarrows}}} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overline{\overline{\mathbb{H}}}) \underset{d_{0*}}{\overset{d_{1*}}{\underset{\longleftarrow i_* }{\rightrightarrows}}} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \overrightarrow{\mathbb{H}}) \underset{d_{0*}}{\overset{d_{1*}}{\underset{\longleftarrow i_* }{\rightrightarrows}}} \mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, \mathbb{H}) \right) \tag{220}$$

by applying $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, -)$ to the diagram (217), where the lower star means action by post-composition. This includes the various induced composition operations m_*, $\overline{m}_*$, $\overline{\overline{m}}_*$, $w_{\ell *}$, w_{r*} and t_*. Because $\mathsf{GrayCat}_{\mathrm{Q}^1}(\mathbb{G}, -)$ by definition preserves limits in the second variable, it takes internal Gray-categories in $\mathsf{GrayCat}_{\mathrm{Q}^1}$ to such in Set, that is, to ordinary Gray-categories. In analogy with our earlier notation we write the compositions on $[\mathbb{G}, \mathbb{H}]$ as $*_n$ where n is the dimension of the incident cell, we use $*$ for the tensor of transformations incident on a functor.

7.1. Theorem. *Given a morphism $F\colon \mathbb{G}' \nrightarrow \mathbb{G}$ in $\mathsf{GrayCat}_{\mathrm{Q}^1}$, the map*

$$F^* = [F, \mathbb{H}]\colon [\mathbb{G}, \mathbb{H}] \longrightarrow [\mathbb{G}', \mathbb{H}]$$

acting by pre-composition is a Gray-functor, that is, a strict morphism.

Proof Assume a situation
$$\begin{tikzcd} \mathbb{G}' \arrow[r, "F", "\mid"{marking}] & \mathbb{G} \arrow[r, bend left=60, "G", "\mid"{marking}, ""{name=U, below}] \arrow[r, "H"{description}, ""{name=M}] \arrow[r, bend right=60, "K"', "\mid"{marking}, ""{name=D, above}] & \mathbb{H} \arrow[from=U, to=M, Rightarrow, "\alpha"] \arrow[from=M, to=D, Rightarrow, "\beta"] \end{tikzcd}$$
then we have

$$\begin{aligned} F^*(\beta *_0 \alpha) = (\beta *_0 \alpha)F = m\langle \beta, \alpha \rangle F \\ = m\langle \beta F, \alpha F \rangle = (\beta F) *_0 (\alpha F) = (F^* \beta) *_0 (F^* \alpha)\,. \end{aligned} \tag{221}$$

Also, for identity transformations we have:

$$F^* \mathrm{id}_G = iGF = \mathrm{id}_{GF}\,, \tag{222}$$

hence F^* is a functor. By the same reasoning the higher operations including the tensor, are preserved as well. □

7.2. Remark. *This way $[-, \mathbb{H}]\colon \mathsf{GrayCat}_{\mathrm{Q}^1} \longrightarrow \mathsf{GrayCat}_{\mathrm{Q}^1}$ is an endofunctor for each $\mathbb{H}$.*

7.3. Remark. *The Gray-category $[\mathbb{G}, \mathbb{H}]$ is a Gray-groupoid if $\mathbb{H}$ is one.*

We have not yet fully developed the techniques to proove the following conjecture, but we have reason to believe it to be true.

7.4. Conjecture. *For a morphism $G\colon \mathbb{H} \longrightarrow \mathbb{H}'$ in $\mathsf{GrayCat}_{\mathrm{Q}^1}$, the map*

$$G_* = [\mathbb{G}, G]\colon [\mathbb{G}, \mathbb{H}] \longrightarrow [\mathbb{G}, \mathbb{H}']$$

is a Gray-functor.

A. An Internal Gray-category

We summarize here the axioms of Gray-categories in an internal fashion, that is, using diagrams in a category with pullbacks. We follow the enumeration given in [Crans 1999, section 2]. We verify that our construction satisfies these axioms.

A.1. Definition. *A* Gray**-category** *is a category enriched in the monoidal category* Gray *if 2-categories with the* Gray*-tensor product.*

Explicitly, if Gray was internal to a category with limits C, then we get a notion of Gray-categories internal to C, which is given by the following data:

- a reflexive 3-globular object

$$\begin{tikzcd} C_3 \arrow[r, shift left=2, "s_2"] \arrow[r, shift right=2, "t_2"'] & C_2 \arrow[l, "\mathrm{id}_2"' description] \arrow[r, shift left=2, "s_1"] \arrow[r, shift right=2, "t_1"'] & C_1 \arrow[l, "\mathrm{id}_1"' description] \arrow[r, shift left=2, "s_0"] \arrow[r, shift right=2, "t_0"'] & C_0 \arrow[l, "\mathrm{id}_0"' description] \end{tikzcd} \tag{223}$$

  globularity means

$$s_n s_{n+1} = s_n t_{n+1} \tag{224}$$
$$t_n s_{n+1} = t_n t_{n+1} \tag{225}$$

  so by abuse of notation we shall write

$$s_n = s_n s_{n+1} = s_n t_{n+1} \tag{226}$$
$$t_n = t_n s_{n+1} = t_n t_{n+1}\,. \tag{227}$$

  Reflexive means

$$C_n = s_n \mathrm{id}_n = t_n \mathrm{id}_n\,. \tag{228}$$

  This already captures condition [Crans 1999, 2.3(i)].

- vertical composition operations:

$$\begin{tikzcd} & & C_{n+1} \times_{s_n,t_n} C_{n+1} \arrow[dl] \arrow[d, "\#_0"] \arrow[dr] & & \\ & C_{n+1} \arrow[dl, "t_n"'] \arrow[dr, "s_n"] & C_{n+1} \arrow[dll, "t_n"'] \arrow[drr, "s_n"] & C_{n+1} \arrow[dl, "t_n"'] \arrow[dr, "s_n"] & \\ C_n & & C_n & & C_n \end{tikzcd} \tag{229}$$

  such that each

$$(C_n, C_{n+1}, \#_n, s_n, t_n, \mathrm{id}_n) \tag{230}$$

  is a category.

- compatible, unital, associative left and right actions of C_2 on $C_3 \rightrightarrows C_2$, that is, maps $\#_1 \colon C_2 \times_{s_1,t_1} C_3 \longrightarrow C_3$ and $\#_1 \colon C_3 \times_{s_1,t_1} C_2 \longrightarrow C_3$, that form internal functors as

follows:

$$
\begin{tikzcd}
(C_2 \times_{s_1,t_1} C_3) \times_{C_2\times s_2, C_2\times t_2} (C_2 \times_{s_1,t_1} C_3) \arrow[r, "\#_1\times\#_1"] \arrow[d, "C_2\times\#_2"'] & C_3 \times_{s_2,t_2} C_3 \arrow[d, "\#_2"] \\
C_2 \times_{s_1,t_1} C_3 \arrow[r, "\#_1"] \arrow[d, shift right=2, "C_2\times s_2"'] \arrow[d, shift left=2, "C_2\times t_2"] & C_3 \arrow[d, shift right=2, "s_2"'] \arrow[d, shift left=2, "t_2"] \\
C_2 \times_{s_1,t_1} C_2 \arrow[r, "\#_1"'] \arrow[u, "C_2\times \mathrm{id}_2" description] & C_2 \arrow[u, "\mathrm{id}_2" description]
\end{tikzcd}
\tag{231}
$$

and

$$
\begin{tikzcd}
(C_3 \times_{s_1,t_1} C_2) \times_{s_2\times C_2, t_2\times C_2} (C_3 \times_{s_1,t_1} C_2) \arrow[r, "\#_1\times\#_1"] \arrow[d, "\#_2\times C_2"'] & C_3 \times_{s_2,t_2} C_3 \arrow[d, "\#_2"] \\
C_3 \times_{s_1,t_1} C_2 \arrow[r, "\#_1"] \arrow[d, shift right=2, "s_2\times C_2"'] \arrow[d, shift left=2, "t_2\times C_2"] & C_3 \arrow[d, shift right=2, "s_2"'] \arrow[d, shift left=2, "t_2"] \\
C_2 \times_{s_1,t_1} C_2 \arrow[r, "\#_1"'] \arrow[u, "\mathrm{id}_2\times C_2" description] & C_2 \arrow[u, "\mathrm{id}_2" description]
\end{tikzcd}
\quad . \tag{232}
$$

Unital means

$$
\begin{tikzcd}
C_3 \arrow[r, "\langle \mathrm{id}_1 t_1, C_3\rangle"] \arrow[dr, "C_3"'] & C_2 \times_{s_1,t_1} C_3 \arrow[d, "\#_1"] \\
& C_3
\end{tikzcd}
\tag{233}
$$

and

$$
\begin{tikzcd}
C_3 \arrow[r, "\langle C_3, \mathrm{id}_1 s_1\rangle"] \arrow[dr, "C_3"'] & C_3 \times_{s_1,t_1} C_2 \arrow[d, "\#_1"] \\
& C_3
\end{tikzcd}
\quad , \tag{234}
$$

and left and right associativity means

$$
\begin{tikzcd}
C_2 \times_{s_1,t_1} C_2 \times_{s_1,t_1} C_3 \arrow[r, "(\#_1)\times C_3"] \arrow[d, "C_2\times(\#_1)"'] & C_2 \times_{s_1,t_1} C_3 \arrow[d, "\#_1"] \\
C_2 \times_{s_1,t_1} C_3 \arrow[r, "\#_1"'] & C_3
\end{tikzcd}
\tag{235}
$$

and

$$
\begin{tikzcd}
C_3 \times_{s_1,t_1} C_2 \times_{s_1,t_1} C_2 \arrow[r, "(\#_1)\times C_2"] \arrow[d, "C_3\times(\#_1)"'] & C_3 \times_{s_1,t_1} C_2 \arrow[d, "\#_1"] \\
C_3 \times_{s_1,t_1} C_2 \arrow[r, "\#_1"'] & C_3
\end{tikzcd}
\quad . \tag{236}
$$

Compatibility means

$$\begin{tikzcd}
C_2 \times_{s_1,t_1} C_3 \times_{s_1,t_1} C_2 \arrow[r, "(\#_1)\times C_2"] \arrow[d, "C_2\times(\#_1)"'] & C_3 \times_{s_1,t_1} C_2 \arrow[d, "\#_1"] \\
C_3 \times_{s_1,t_1} C_2 \arrow[r, "\#_1"'] & C_3
\end{tikzcd} \quad . \tag{237}$$

Furthermore we demand that the horizontal whiskers $\#_1$ of 3-cells by 2-cells along 1-cells, and vertical composition $\#_2$ of 3-cells along 2-cells define a unique horizontal composition of 3-cells along a 1-cell, that is,

$$\begin{tikzcd}
C_3 \times_{s_1,t_1} C_3 \arrow[r, "\langle(\#_1)(t_2\times C_3),(\#_1)(C_3\times s_2)\rangle"] \arrow[d, "\langle(\#_1)(C_3\times t_2),(\#_1)(s_2\times C_3)\rangle"'] & C_3 \times_{s_2,t_2} C_3 \arrow[d, "\#_2"] \\
C_3 \times_{s_2,t_2} C_3 \arrow[r, "\#_2"'] & C_3
\end{tikzcd} \quad . \tag{238}$$

This point together with the previous one captures [Crans 1999, 2.4(ii)].

- Furthermore, 2-functorial, compatible, unital, associative left and right actions of C_1 on the 2-category $C_3 \rightrightarrows C_2 \rightrightarrows C_1$, given by maps $\#_0\colon C_1 \times_{s_0,t_0} C_2 \longrightarrow C_2$, $\#_0\colon C_2 \times_{s_0,t_0} C_1 \longrightarrow C_2$, $\#_0\colon C_1 \times_{s_0,t_0} C_3 \longrightarrow C_3$, and $\#_0\colon C_3 \times_{s_0,t_0} C_1 \longrightarrow C_3$. In detail this means, left and right functoriality with respect to 2-cells

$$\begin{tikzcd}
(C_1 \times_{s_0,t_0} C_2) \times_{C_1\times s_1, C_1\times t_1} (C_1 \times_{s_0,t_0} C_2) \arrow[r, "\#_0\times\#_0"] \arrow[d, "C_1\times\#_1"'] & C_2 \times_{s_1,t_1} C_2 \arrow[d, "\#_1"] \\
C_1 \times_{s_0,t_0} C_2 \arrow[r, "\#_0"] \arrow[d, shift right=2, "C_1\times s_1"'] \arrow[d, shift left=2, "C_1\times t_1"] & C_2 \arrow[d, shift right=2, "s_1"'] \arrow[d, shift left=2, "t_1"] \\
C_1 \times_{s_0,t_0} C_1 \arrow[r, "\#_0"'] \arrow[u, "C_1\times \mathrm{id}_1"] & C_1 \arrow[u, "\mathrm{id}_1"]
\end{tikzcd} \quad , \tag{239}$$

$$
\begin{tikzcd}
(C_2 \times_{s_0,t_0} C_1) \times_{s_1 \times C_1, t_1 \times C_1} (C_2 \times_{s_0,t_0} C_1) \arrow[r, "\#_0 \times \#_0"] \arrow[d, "\#_1 \times C_1"'] & C_2 \times_{s_1,t_1} C_2 \arrow[d, "\#_1"] \\
C_2 \times_{s_0,t_0} C_1 \arrow[r, "\#_0"] \arrow[d, shift right=2, "s_1 \times C_1"'] \arrow[d, shift left=2, "t_1 \times C_1"] & C_2 \arrow[d, shift right=2, "s_1"'] \arrow[d, shift left=2, "t_1"] \\
C_1 \times_{s_0,t_0} C_1 \arrow[r, "\#_0"'] \arrow[u, "\mathrm{id}_1 \times C_1" description] & C_1 \arrow[u, "\mathrm{id}_1" description]
\end{tikzcd}
\quad ; \tag{240}
$$

left and right functoriality with respect to 3-cells

$$
\begin{tikzcd}
(C_1 \times_{s_0,t_0} C_3) \times_{C_1 \times s_2, C_1 \times t_2} (C_1 \times_{s_0,t_0} C_3) \arrow[r, "\#_0 \times \#_0"] \arrow[d, "C_1 \times \#_2"'] & C_3 \times_{s_2,t_2} C_3 \arrow[d, "\#_2"] \\
C_1 \times_{s_0,t_0} C_3 \arrow[r, "\#_0"] \arrow[d, shift right=2, "C_1 \times s_2"'] \arrow[d, shift left=2, "C_1 \times t_2"] & C_3 \arrow[d, shift right=2, "s_2"'] \arrow[d, shift left=2, "t_2"] \\
C_1 \times_{s_0,t_0} C_2 \arrow[r, "\#_0"'] \arrow[u, "C_1 \times \mathrm{id}_2" description] & C_2 \arrow[u, "\mathrm{id}_2" description]
\end{tikzcd}
\quad , \tag{241}
$$

$$
\begin{tikzcd}
(C_3 \times_{s_0,t_0} C_1) \times_{s_2 \times C_1, t_2 \times C_1} (C_3 \times_{s_0,t_0} C_1) \arrow[r, "\#_0 \times \#_0"] \arrow[d, "\#_2 \times C_1"'] & C_3 \times_{s_2,t_2} C_3 \arrow[d, "\#_2"] \\
C_3 \times_{s_0,t_0} C_1 \arrow[r, "\#_0"] \arrow[d, shift right=2, "s_1 \times C_1"'] \arrow[d, shift left=2, "t_1 \times C_1"] & C_2 \arrow[d, shift right=2, "s_1"'] \arrow[d, shift left=2, "t_1"] \\
C_2 \times_{s_0,t_0} C_1 \arrow[r, "\#_0"'] \arrow[u, "\mathrm{id}_1 \times C_1" description] & C_2 \arrow[u, "\mathrm{id}_2" description]
\end{tikzcd}
\quad . \tag{242}
$$

Unitality of the $\#_0$ whiskering actions means

$$\begin{tikzcd} C_2 \arrow[r, "\langle \mathrm{id}_0 t_0, C_2\rangle"] \arrow[dr, "C_2"'] & C_1 \times_{s_0,t_0} C_2 \arrow[d, "\#_0"] \\ & C_2 \end{tikzcd} , \tag{243}$$

$$\begin{tikzcd} C_2 \arrow[r, "\langle C_2, \mathrm{id}_0 s_0\rangle"] \arrow[dr, "C_2"'] & C_2 \times_{s_0,t_0} C_1 \arrow[d, "\#_0"] \\ & C_2 \end{tikzcd} , \tag{244}$$

similarly for the action of 1-cells on 3-cells,

$$\begin{tikzcd} C_3 \arrow[r, "\langle \mathrm{id}_0 t_0, C_3\rangle"] \arrow[dr, "C_3"'] & C_1 \times_{s_0,t_0} C_3 \arrow[d, "\#_0"] \\ & C_3 \end{tikzcd} , \tag{245}$$

$$\begin{tikzcd} C_3 \arrow[r, "\langle C_3, \mathrm{id}_0 s_0\rangle"] \arrow[dr, "C_3"'] & C_3 \times_{s_0,t_0} C_1 \arrow[d, "\#_0"] \\ & C_3 \end{tikzcd} . \tag{246}$$

Left and right associativity as well as compatibility mean

$$\begin{tikzcd} C_1 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_2 \arrow[r, "(\#_0)\times C_2"] \arrow[d, "C_1\times(\#_0)"'] & C_1 \times_{s_0,t_0} C_2 \arrow[d, "\#_0"] \\ C_1 \times_{s_1,t_1} C_2 \arrow[r, "\#_0"'] & C_2 \end{tikzcd} , \tag{247}$$

$$\begin{tikzcd} C_2 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_1 \arrow[r, "(\#_0)\times C_1"] \arrow[d, "C_2\times(\#_0)"'] & C_2 \times_{s_0,t_0} C_1 \arrow[d, "\#_0"] \\ C_2 \times_{s_0,t_0} C_2 \arrow[r, "\#_0"'] & C_2 \end{tikzcd} , \tag{248}$$

$$\begin{tikzcd} C_1 \times_{s_0,t_0} C_2 \times_{s_0,t_0} C_1 \arrow[r, "(\#_0)\times C_1"] \arrow[d, "C_1\times(\#_0)"'] & C_2 \times_{s_0,t_0} C_1 \arrow[d, "\#_0"] \\ C_1 \times_{s_0,t_0} C_2 \arrow[r, "\#_0"'] & C_2 \end{tikzcd} , \tag{249}$$

$$\begin{tikzcd} C_1 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_3 \arrow[r, "(\#_0)\times C_3"] \arrow[d, "C_1\times(\#_0)"'] & C_1 \times_{s_0,t_0} C_3 \arrow[d, "\#_0"] \\ C_1 \times_{s_1,t_1} C_3 \arrow[r, "\#_0"'] & C_3 \end{tikzcd} , \tag{250}$$

$$\begin{tikzcd} C_3 \times_{s_0,t_0} C_1 \times_{s_0,t_0} C_1 \arrow[r, "(\#_0)\times C_1"] \arrow[d, "C_3\times(\#_0)"'] & C_3 \times_{s_0,t_0} C_1 \arrow[d, "\#_0"] \\ C_3 \times_{s_0,t_0} C_1 \arrow[r, "\#_0"'] & C_3 \end{tikzcd} , \tag{251}$$

$$\begin{tikzcd} C_1 \times_{s_0,t_0} C_3 \times_{s_0,t_0} C_1 \arrow[r, "(\#_0)\times C_1"] \arrow[d, "C_1\times(\#_0)"'] & C_3 \times_{s_0,t_0} C_1 \arrow[d, "\#_0"] \\ C_1 \times_{s_0,t_0} C_3 \arrow[r, "\#_0"'] & C_3 \end{tikzcd} . \tag{252}$$

This covers conditions Crans [1999, 2.4(iii)&(iv)]. The equations (243) and (246) capture Crans [1999, 2.4(viii)]

- a map $\otimes\colon C_2 \times_{s_0,t_0} C_2 \longrightarrow C_3$ such that

$$\begin{tikzcd} C_2 \times_{s_0,t_0} C_2 \arrow[rr, "\langle(\#_0)(t_1\times C_2),(\#_0)(C_2\times s_1)\rangle"] \arrow[dd, "\langle(\#_0)(C_2\times t_1),(\#_0)(s_1\times C_2)\rangle"'] \arrow[dr, "\otimes"] & & C_2 \times_{s_1,t_1} C_2 \arrow[dd, "\#_1"] \\ & \dot{C}_3 \arrow[dr, shift left, "t_2"] \arrow[dr, shift right, "s_2"'] & \\ C_2 \times_{s_1,t_1} C_2 \arrow[rr, "\#_1"'] & & C_2 \end{tikzcd} , \tag{253}$$

where $\dot{C}_3$ is the object of invertible 3-cells.c This expresses condition [Crans 1999, 2.4(v)].

- Abbreviating

$$
\begin{aligned}
\triangleright &=(\#_1)\langle(\#_0)(t_1 \times C_2),(\#_0)(C_2 \times s_1)\rangle && (254)\\
\triangleleft &=(\#_1)\langle(\#_0)(C_2 \times t_1),(\#_0)(s_1 \times C_2)\rangle && (255)\\
\triangleright_\ell &=(\#_1)\langle(\#_0)(t_1 \times C_3),(\#_0)(C_2 \times s_1)\rangle && (256)\\
\triangleleft_\ell &=(\#_1)\langle(\#_0)(C_2 \times t_1),(\#_0)(s_1 \times C_3)\rangle && (257)\\
\triangleright_r &=(\#_1)\langle(\#_0)(t_1 \times C_2),(\#_0)(C_3 \times s_1)\rangle && (258)\\
\triangleleft_r &=(\#_1)\langle(\#_0)(C_3 \times t_1),(\#_0)(s_1 \times C_2)\rangle && (259)
\end{aligned}
$$

we require $\otimes$ to have the following naturality properties

$$
\begin{tikzcd}
C_3 \times_{s_0,t_0} C_2 \arrow[r, "\langle(\triangleright_r),\otimes(s_2\times C_2)\rangle"] \arrow[d, "\langle\otimes(t_2\times C_2),(\triangleleft_r)\rangle"'] & C_3 \times_{s_2,t_2} C_3 \arrow[d, "\#_2"] \\
C_3 \times_{s_2,t_2} C_3 \arrow[r, "\#_2"'] & C_3
\end{tikzcd}
\tag{260}
$$

and

$$
\begin{tikzcd}
C_2 \times_{s_0,t_0} C_3 \arrow[r, "\langle(\triangleright_\ell),\otimes(C_2\times s_2)\rangle"] \arrow[d, "\langle\otimes(C_2\times t_2),(\triangleleft_\ell)\rangle"'] & C_3 \times_{s_2,t_2} C_3 \arrow[d, "\#_2"] \\
C_3 \times_{s_2,t_2} C_3 \arrow[r, "\#_2"'] & C_3
\end{tikzcd}
\quad .
\tag{261}
$$

This expresses condition [Crans 1999, 2.4(vi)].

- Functoriality of the tensor. [Crans 1999, (vii)]

$$
\begin{tikzcd}
C_2 \times_{s_0,s_t} (C_2 \times_{s_1,t_2} C_2) \arrow[r, "C_2\times(\#_1)"] \arrow[d, "\langle\otimes(C_2\times p_1),\otimes(C_2\times p_2)\rangle"'] & C_2 \times_{s_0,t_0} C_2 \arrow[d, "\otimes"] \\
\dot{C}_3 \times_{s_2,t_2} \dot{C}_3 \arrow[r, "\#_2"'] & \dot{C}_3
\end{tikzcd}
\tag{262}
$$

$$
\begin{tikzcd}
(C_2 \times_{s_1,t_2} C_2) \times_{s_0,s_t} C_2 \arrow[r, "(\#_1)\times C_2"] \arrow[d, "\langle\otimes(p_1\times C_2),\otimes(p_1\times C_2)\rangle"'] & C_2 \times_{s_0,t_0} C_2 \arrow[d, "\otimes"] \\
\dot{C}_3 \times_{s_2,t_2} \dot{C}_3 \arrow[r, "\#_2"'] & \dot{C}_3
\end{tikzcd}
\tag{263}
$$

- Associativity of the $\#_0$ compositions [Crans 1999, (ix)]

B. Adjunctions

We can embed the ideas developed in section 3 in a more global picture. The functor $\mathrm{Q}^1 \colon \mathsf{GrayCat} \longrightarrow \mathsf{GrayCat}$ is part of the following adjunction of fibered categories:

$$\begin{tikzcd} F^*(\mathsf{GrayCat}) \arrow[r, bend left, "(\_)_1{}^*(F)"] \arrow[r, phantom, "\perp"] \arrow[d, "F^*((\_)_1)"'] & \mathsf{GrayCat} \arrow[l, bend left, "\underline{U}"] \arrow[d, "(\_)_1"] \\ \mathsf{RGrph} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "U"] \end{tikzcd} \tag{264}$$

where F means "free category over a reflexive graph" and U means "underlying reflexive graph of a category", $(\_)_1$ means "underlying category of a Gray-category. According to [Hermida 1999, 4.1] the adjunction $F \dashv U$ lifts canonically to an adjunction $((\_)_1{}^*(F), F) \dashv (\underline{U}, U)$ of fibered categories. Which means in particular that $(\_)_1{}^*(F) \dashv \underline{U}$ is an adjunction and our Q^1 can be defined as $(\_)_1{}^*(F)\underline{U}$.

The objects of $\mathsf{Graph} \times \mathsf{GrayCat}$ might be called 1-free Gray-categories.

We can construct a further resolution which we call Q^2.

B.1. Remark. *Let $P \colon \mathcal{E} \longrightarrow \mathcal{B}$ be a 2-fibration in the sense of Hermida [1999]. Given $u \colon I \longrightarrow PX$ and $u' \colon I' \longrightarrow PX$ for X an object in $\mathcal{E}$; and an equivalence $h \colon I \longrightarrow I'$ such that $u'h = u$. Then the unique filler $\widehat{h}$ over h is an equivalence as well.*

In particular, given the comparison functor $K \colon \mathsf{X}_{FU} \longrightarrow \mathsf{A}$ for the comonad induced by $F \dashv U \colon \mathsf{A} \longrightarrow \mathsf{X}$ lifts to a comparison functor $\widehat{K}$.

B.2. Lemma. *If F is comonadic, then so is $((\_)_1{}^*(F), F)$.*

C. Putting it all together

C.1. Definition. *A* **lax transformation** *$\alpha \colon F \longrightarrow G$ between pseudo-functors $F, G \colon \mathbb{G} \twoheadrightarrow \mathbb{H}$ of Gray-categories is a pseudo-functor $\alpha \colon \mathbb{G} \twoheadrightarrow \overrightarrow{\mathbb{H}}$ such that $d_0\alpha = F$ and $d_1\alpha = G$.*

C.2. Remark. *Using the definition of path spaces in 4.1 and the characterization of pseudo-maps in 3.20 we note for reference that a lax transformation α is given by the following underlying data:*

1. *for each 0-cell x of $\mathbb{G}$ a 1-cell $\alpha_x \colon Fx \longrightarrow Gx$,*
2. *for each 1-cell $f \colon x \longrightarrow y$ of $\mathbb{G}$ a 2-cell*

$$\begin{tikzcd} Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\alpha_f"] \\ Fy \arrow[r, "\alpha_y"'] & Gy \end{tikzcd} \tag{265}$$

3. for each 2-cell $g\colon f \longrightarrow f'$ of $\mathbb{G}$ a 3-cell of $\mathbb{H}$

$$
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=60, "Ff'"'] \arrow[d, phantom, shift right=4ex, "\overset{\Leftarrow}{Fg}"] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow\alpha_f"] & & Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gx \arrow[d, "Gf'"'] \arrow[d, bend left=60, "Gf"] \arrow[d, phantom, shift left=4ex, "\overset{\Leftarrow}{Gg}"] \arrow[dl, phantom, "\Swarrow\alpha_{f'}"] \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[rr, phantom, "\overset{\alpha_g}{\Rrightarrow}"] & & Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}
\tag{266}
$$

4. for each pair of composable 1-cells $f\colon x \longrightarrow y$, $f'\colon y \longrightarrow z$ an invertible 3-cell

$$
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[dd, bend right=70, "F(f'\#_0 f)"'] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow\alpha_f"] & & Fx \arrow[r, "\alpha_x"] \arrow[dd, "F(f'\#_0 f)"'] & Gx \arrow[dd, "G(f'\#_0 f)"] \arrow[dr, "Gf"] \arrow[ddl, phantom, "\Swarrow\alpha_{f'\#_0 f}"] & \\
Fy \arrow[r, "\alpha_y"'] \arrow[d, "Ff'"] \arrow[l, phantom, "\overset{\Longleftarrow}{F^2_{f',f}}"] & Gy \arrow[d, "Gf'"] \arrow[dl, phantom, "\Swarrow\alpha_{f'}"] \arrow[rr, phantom, "\overset{\alpha^2_{f',f}}{\Rrightarrow}"] & & & \arrow[r, phantom, "\overset{G^2_{f',f}}{\Leftarrow}"] & Gy \arrow[dl, "Gf'"] \\
Fz \arrow[r, "\alpha_z"'] & Gz & & Fz \arrow[r, "\alpha_z"'] & Gz &
\end{tikzcd}
\ .
\tag{267}
$$

Furthermore, these data have to satisfy the following equations:

1. On identities of 0-cells:

$$
\alpha_{\mathrm{id}_x} = \mathrm{id}_{\alpha_x}
\tag{268}
$$

2. for each 3-cell $\Gamma\colon g \longrightarrow g'$ the square of 3-cells in $\mathbb{H}$

$$
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=60, "Ff'"'] \arrow[d, phantom, shift right=4ex, "\overset{\Leftarrow}{Fg}"] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow\alpha_f"] & & Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gx \arrow[d, "Gf'"'] \arrow[d, bend left=60, "Gf"] \arrow[d, phantom, shift left=4ex, "\overset{\Leftarrow}{Gg}"] \arrow[dl, phantom, "\Swarrow\alpha_{f'}"] \\
Fy \arrow[r, "\alpha_y"'] \arrow[ddd, phantom, "\Big\Downarrow (\alpha_y\#_0 F\Gamma)\#_1\alpha_f"] & Gy \arrow[rr, phantom, "\overset{\alpha_g}{\Rrightarrow}"] & & Fy \arrow[r, "\alpha_y"'] & Gy \arrow[ddd, phantom, "\Big\Downarrow \alpha_{f'}\#_1(G\Gamma\#_0\alpha_x)"] \\
& & & & \\
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=60, "Ff'"'] \arrow[d, phantom, shift right=4ex, "\overset{\Leftarrow}{Fg'}"] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow\alpha_f"] & & Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gy \arrow[d, "Gf'"'] \arrow[d, bend left=60, "Gf"] \arrow[d, phantom, shift left=4ex, "\overset{\Leftarrow}{Gg'}"] \arrow[dl, phantom, "\Swarrow\alpha_{f'}"] \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[rr, phantom, "\underset{\alpha_{g'}}{\Rrightarrow}"] & & Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}
\tag{269}
$$

commutes. This condition obviously comes from the definition of 3-cells in the path space.

3. *For every pair $g\colon f \Longrightarrow f', g'\colon f' \Longrightarrow f''$:*

$$
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] \arrow[d, bend right=60, "Ff'"'] \arrow[d, bend right=90, "Ff''"'] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow \alpha_f"] \\
Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}
\overset{(\alpha_y \#_0 Fg')\#_1 \alpha_g}{\Rrightarrow}
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] \arrow[d, bend right=60, "Ff''"'] & Gx \arrow[d, "Gf'"] \arrow[d, bend left=60, "Gf"] \arrow[dl, phantom, "\Swarrow \alpha_{f'}"] \\
Fx \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}
\overset{\alpha_{g'}\#_1(Gg\#_0\alpha_x)}{\Rrightarrow}
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff''"'] & Gx \arrow[d, "Gf''"] \arrow[d, bend left=60, "Gf'"] \arrow[d, bend left=90, "Gf"] \arrow[dl, phantom, "\Swarrow \alpha_{f''}"] \\
Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}
,\qquad \text{with } \alpha_{g'\#_1 g} \text{ from the first to the third} \tag{270}
$$

and for identity 2-cells $\mathrm{id}_f\colon f \Longrightarrow f$ we have an identity 3-cell

$$\alpha_{\mathrm{id}_f} = \mathrm{id}_{\alpha_f}\,. \tag{271}$$

4. *The family of 3-cells has to satisfy a kind of cocycle condition: For a composable triple f, f', f'' of 1-cells α^2 has to satisfy equation (272). furthermore, α^2 has to satisfy the normalization condition:*

$$\alpha^2_{f',f} = \begin{cases} \mathrm{id}_{\alpha_{f'}} & \text{if } f' = \mathrm{id}_y \\ \mathrm{id}_{\alpha_f} & \text{if } f = \mathrm{id}_x \end{cases} \tag{273}$$

5. *The family of 3-cells α^2 has to be compatible with left and right whiskering according to (274) and (275).*

*These conditions are derived from ones in the definition of pseudo-*Gray*-functors 3.20. Note how conditions 4, 5, 6 of 3.20 are trivially satisfied for transformations.*

C.3. Definition. *A transformation $\alpha\colon F \longrightarrow G$ where the cocycle α^2 has only trivial components we call a* **stiff transformation**.

C.4. Lemma. *A stiff transformation $\alpha\colon F \longrightarrow G$ with F and G strict* Gray*-functors is a 1-transfor in the sense of [Crans 1999].* □

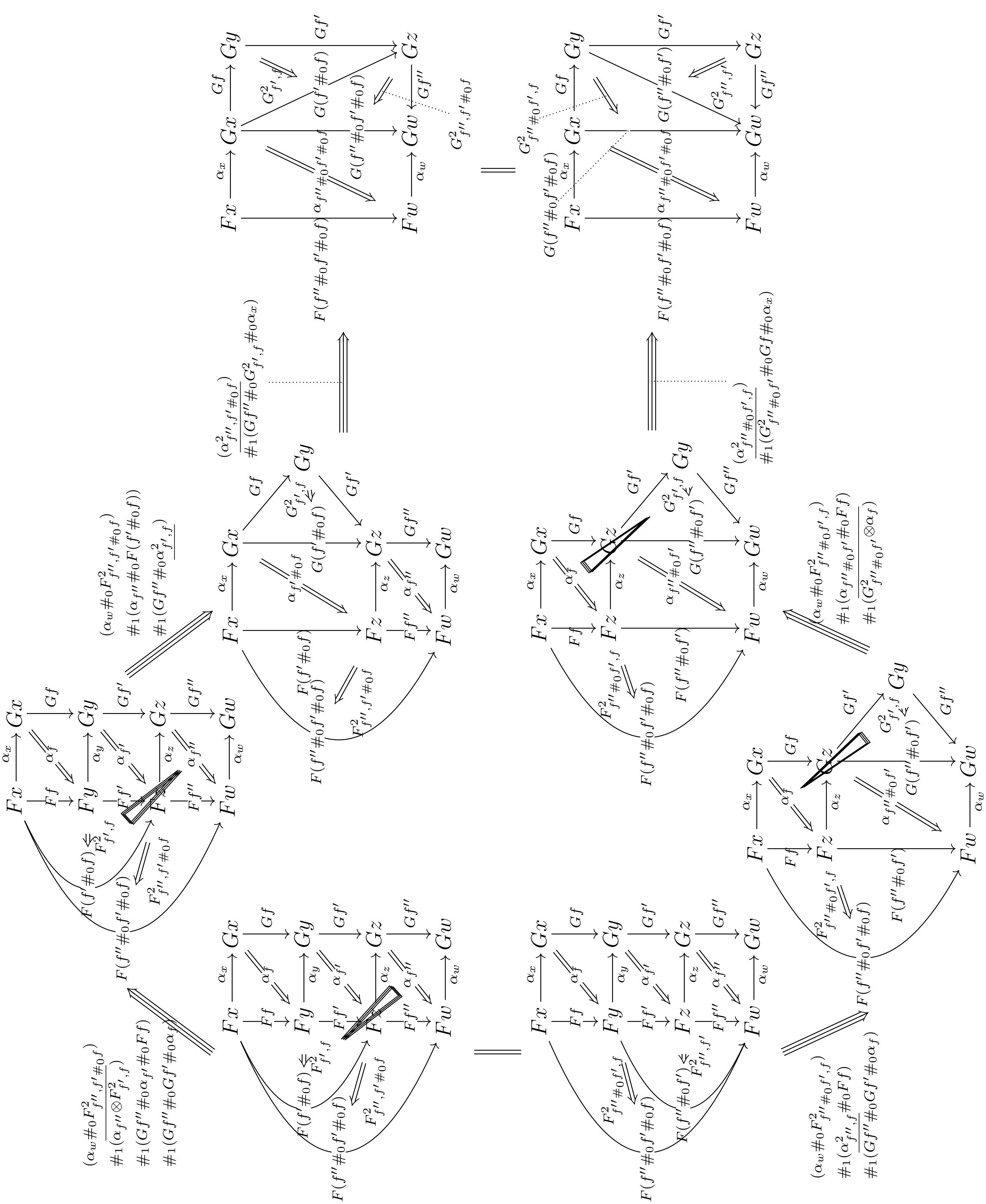
C.5. Remark. *Given two lax-transformations $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ their composite $\beta * \alpha$ given by $m\langle \beta, \alpha\rangle$ and has the following components:*

1. *for each 0-cell x of $\mathbb{G}$ the 1-cell*

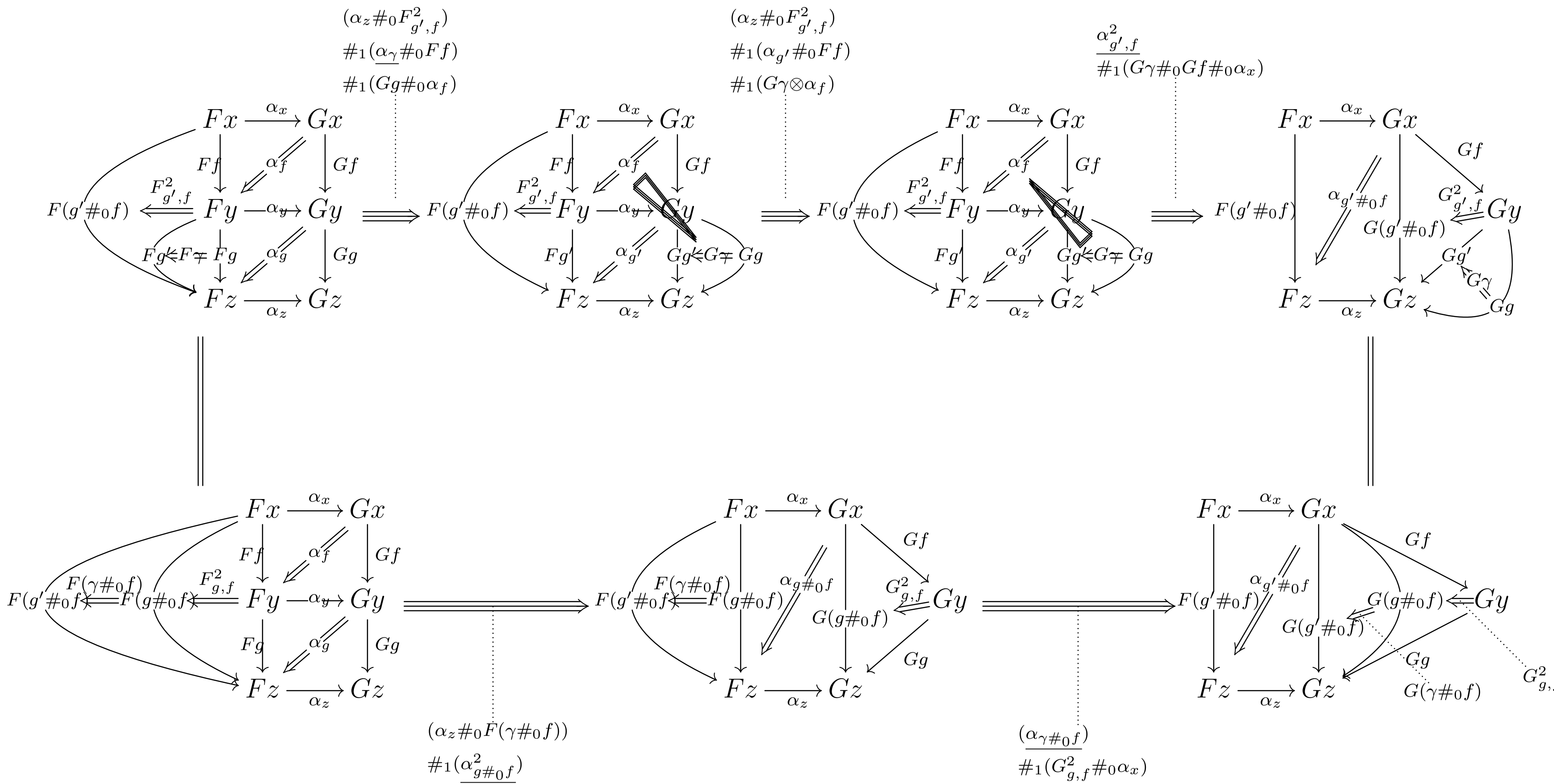
$$Fx \xrightarrow{(\beta*\alpha)_x} Hx \;=\; Fx \xrightarrow{\alpha_x} Gx \xrightarrow{\beta_x} Hx\,, \tag{276}$$

2. *for each 1-cell $f\colon x \longrightarrow y$ of $\mathbb{G}$ the 2-cell*

$$
\begin{tikzcd}
Fx \arrow[r, "(\beta*\alpha)_x"] \arrow[d, "Ff"'] & Hx \arrow[d, "Hf"] \arrow[dl, phantom, "\Swarrow (\beta*\alpha)_f"] \\
Fy \arrow[r, "(\beta*\alpha)_y"'] & Hy
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow \alpha_f"] & Hx \arrow[d, "Hf"] \arrow[dl, phantom, "\Swarrow \beta_f"] \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[r, "\beta_y"'] & Hy
\end{tikzcd}
\tag{277}
$$

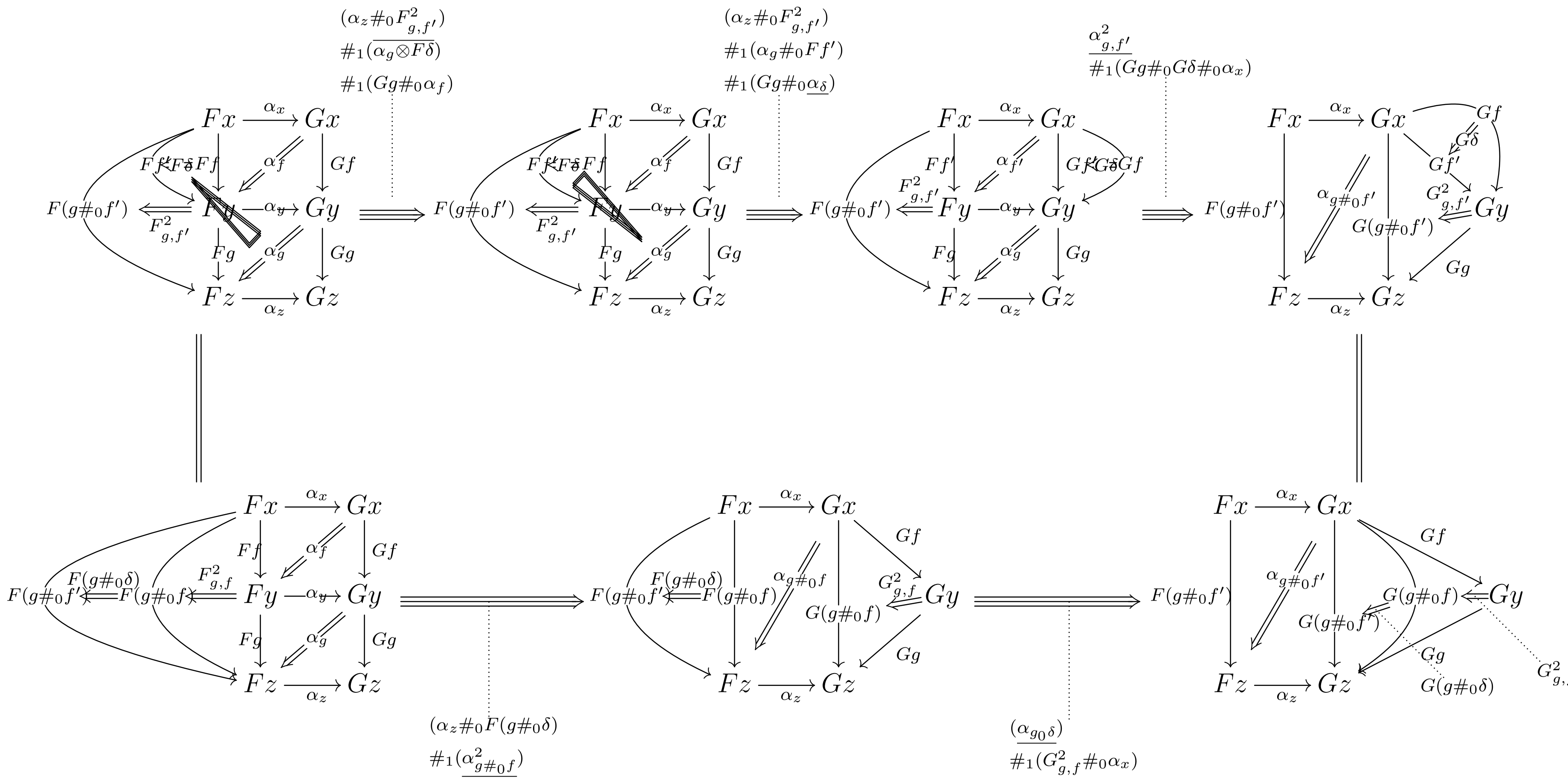


(272)



(274)

Compatibility of the cocycle α^2 with left whiskers $\gamma \#_0 f$.



$$(275)$$

Compatibility of the cocycle α^2 with right whiskers $g\#_0\delta$.

3. *for each 2-cell $g\colon f \longrightarrow f'$ of $\mathbb{G}$ the 3-cell of $\mathbb{H}$ shown in* (278)

4. *for each pair of composable 1-cells $f\colon x \longrightarrow y$, $f'\colon y \longrightarrow z$ a 3-cell shown in* (279)

C.6. Definition. *Assuming α and β are as in definition C.1 and F and G are pseudo-functors $\mathbb{G} \nrightarrow \mathbb{H}$, a* **modification** $A\colon \alpha \longrightarrow \beta\colon F \longrightarrow G$ *is a pseudo-functor $A\colon \mathbb{G} \nrightarrow \overline{\overline{\overline{\mathbb{H}}}}$, such that $d_0 A = \alpha$ and $d_1 A = \beta$.*

C.7. Remark. *A modification $A\colon \alpha \longrightarrow \beta$ according to C.6 and 3.20 is given by the following data:*

1. *For every 0-cell x in $\mathbb{G}$ a 2-cell*

$$\begin{tikzcd} Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[r, bend right=50, "\beta_x"'] \arrow[r, phantom, "\Downarrow A_x"] & Gx \end{tikzcd} \tag{280}$$

2. *For every 1-cell $f\colon x \longrightarrow y$ a 3-cell in $\mathbb{H}$*

$$\begin{tikzcd} Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[r, bend right=50, "\beta_x"'] \arrow[r, phantom, "\Downarrow A_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, bend right=50, "\beta_y"'] \arrow[ur, phantom, "\swarrow \beta_f"] & Gy \end{tikzcd} \overset{A_f}{\Rrightarrow} \begin{tikzcd} Fx \arrow[r, bend left=50, "\alpha_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\swarrow \alpha_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, bend left=50, "\alpha_y"] \arrow[r, bend right=50, "\beta_y"'] \arrow[r, phantom, "\Downarrow A_y"] & Gy \end{tikzcd} \tag{281}$$

This data has to satisfy the following conditions:

1. *Units are preserved:*

$$A_{\mathrm{id}_x} = \mathrm{id}_{A_x} \tag{282}$$

2. *Compatibility with the cocycles of F, G, α, β according to (283)*

3. *For 2-cells $g\colon f \Longrightarrow f'$ in $\mathbb{G}$ the images under F and G as well the data of A, α and β are compatible as shown in (284)*

C.8. Lemma. *A transformation $A\colon \alpha \longrightarrow \beta$ where $\alpha, \beta\colon F \longrightarrow G$ are stiff and F, G are strict is a 2-transfor in the sense of [Crans 1999].* □

C.9. Definition. *Given modifications $A, B\colon \alpha \longrightarrow \beta$ a* **perturbation** *is a pseudo-*Gray*-functor $\sigma\colon \mathbb{G} \nrightarrow \overline{\overline{\overline{\overline{\mathbb{H}}}}}$ such that $d_0\sigma = A$ and $d_1\sigma = B$.*

$$
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=80, "Ff'"'] \arrow[d, phantom, bend right=40, "\overset{\Leftarrow}{Fg}"] \arrow[dr, phantom, "\Swarrow (\beta*\alpha)_f"] & Hx \arrow[d, "Hf"] \\
Fy \arrow[r, "(\beta*\alpha)_y"'] & Hy
\end{tikzcd}
\overset{(\beta*\alpha)_g}{\Rrightarrow}
\begin{tikzcd}
Fx \arrow[r, "(\beta*\alpha)_x"] \arrow[d, "Ff'"'] \arrow[dr, phantom, "\Swarrow (\beta*\alpha)_{f'}"] & Hx \arrow[d, "Hf'"] \arrow[d, bend left=80, "Hf"] \arrow[d, phantom, bend left=40, "\overset{\Leftarrow}{Hg}"] \\
Fy \arrow[r, "(\beta*\alpha)_y"'] & Hy
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[d, bend right=80, "Ff'"'] \arrow[d, phantom, bend right=40, "\overset{\Leftarrow}{Fg}"] \arrow[dr, phantom, "\Swarrow \alpha_f"] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf"] \arrow[dr, phantom, "\Swarrow \beta_f"] & Hx \arrow[d, "Hf"] \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[r, "\beta_y"'] & Hy
\end{tikzcd}
\overset{\substack{(\beta_y \#_0 \alpha_g) \\ \#_1(\beta_f \#_0 \alpha_x)}}{\Rrightarrow}
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] & Gx \arrow[r, "\beta_x"] \arrow[d, bend right, "Gf'"'] \arrow[d, bend left, "Gf"] \arrow[d, phantom, "\overset{\Leftarrow}{Gg}"] & Hx \arrow[d, "Hf"] \\
Fy \arrow[r, "\alpha_y"'] \arrow[ur, phantom, "\Swarrow \alpha_{f'}"] & Gy \arrow[r, "\beta_y"'] \arrow[ur, phantom, "\Swarrow \beta_f"] & Hy
\end{tikzcd}
\overset{\substack{(\beta_y \#_0 \alpha_{f'}) \\ \#_1(\beta_g \#_0 \alpha_x)}}{\Rrightarrow}
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'"'] \arrow[dr, phantom, "\Swarrow \alpha_{f'}"] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf'"] \arrow[dr, phantom, "\Swarrow \beta_{f'}"] & Hx \arrow[d, "Hf'"] \arrow[d, bend left=80, "Hf"] \arrow[d, phantom, bend left=40, "\overset{\Leftarrow}{Hg}"] \\
Fy \arrow[r, "\alpha_y"'] & Gy \arrow[r, "\beta_y"'] & Hy
\end{tikzcd}
\tag{278}
$$

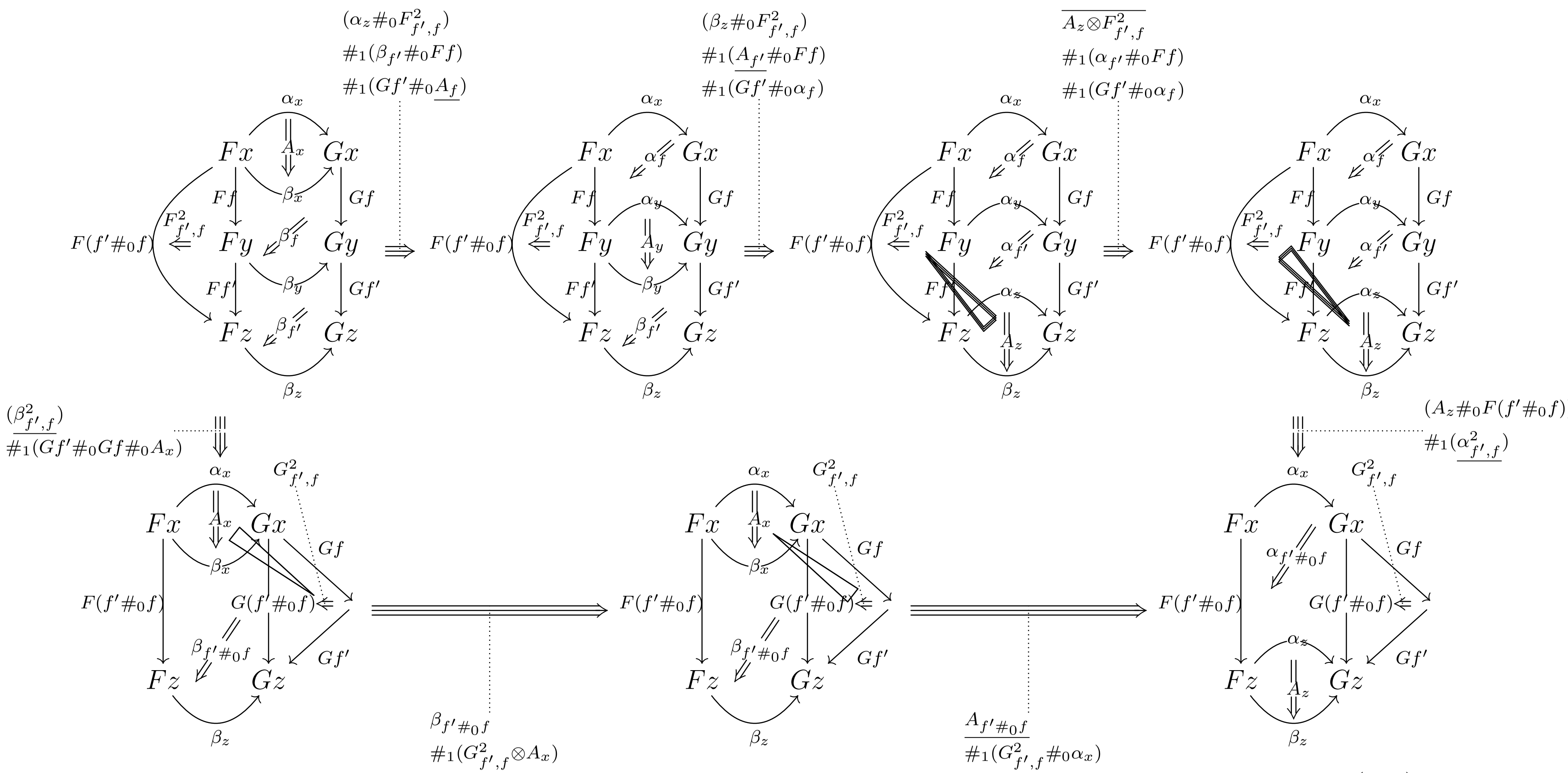
$$
\begin{tikzcd}
Fx \arrow[r, "(\beta*\alpha)_x"] \arrow[d, "Ff"] \arrow[dd, bend right=60, "F(f'\#_0 f)"'] \arrow[dd, phantom, bend right=30, "\Leftarrow F^2_{f',f}"] & Hx \arrow[d, "Hf"] \arrow[dl, phantom, "\Swarrow (\beta*\alpha)_f"] \\
Fy \arrow[r, "(\beta*\alpha)_y"] \arrow[d, "Ff'"] & Hy \arrow[d, "Hf'"] \arrow[dl, phantom, "\Swarrow (\beta*\alpha)_{f'}"] \\
Fz \arrow[r, "(\beta*\alpha)_z"'] & = Hz
\end{tikzcd}
\overset{(\beta*\alpha)^2_{f',f}}{\Rrightarrow}
\begin{tikzcd}
Fx \arrow[r, "(\beta_\alpha)_x"] \arrow[d, "F(f'\#_0 f)"'] & Hx \arrow[d, "H(f'\#_0 f)"] \arrow[d, bend left=60, "Hf'\#_0 Hf"] \arrow[d, phantom, bend left=30, "\Leftarrow H^2_{f',f}"] \arrow[dl, phantom, "\Swarrow (\beta*\alpha)_{f'\#_0 f}"] \\
Fy \arrow[r, "(\beta*\alpha)_z"'] & Hy
\end{tikzcd}
$$

$$
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"] \arrow[dd, bend right=60, "F(f'\#_0 f)"'] \arrow[dd, phantom, bend right=30, "\Leftarrow F^2_{f',f}"] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf"] \arrow[dl, phantom, "\Swarrow \alpha_f"] & Hx \arrow[d, "Hf"] \arrow[dl, phantom, "\Swarrow \beta_f"] \\
Fy \arrow[r, "\alpha_y"] \arrow[d, "Ff'"] & Gy \arrow[r, "\beta_y"] \arrow[d, "Gf'"] \arrow[dl, phantom, "\Swarrow \alpha_{f'}"] & Hy \arrow[d, "Hf'"] \arrow[dl, phantom, "\Swarrow \beta_{f'}"] \\
Fz \arrow[r, "\alpha_z"'] & Gz \arrow[r, "\beta_z"'] & Hz
\end{tikzcd}
\xRrightarrow{\substack{(\beta_z\#_0\alpha_z\#_0 F^2_{f',f}) \\ \#_1(\beta_z\#_0\alpha_{f'}\#_0 Ff) \\ \#_1(\beta_{f'}\otimes\alpha_f) \\ \#_1(Hf'\#_0\beta_f\#_0\alpha_x)}}
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff'\#_0 Ff"] \arrow[d, bend right=60, "F(f'\#_0 f)"'] \arrow[d, phantom, bend right=30, "\Leftarrow F^2_{f',f}"] & Gx \arrow[r, "\beta_x"] \arrow[d, "Gf'\#_0 Gf"] \arrow[dl, phantom, "\Swarrow \substack{(\alpha_{f'}\#_0 Ff) \\ \#_1(Gf'\#_0\alpha_f)}"] & Hx \arrow[d, "Hf'\#_0 Hf"] \arrow[dl, phantom, "\Swarrow \substack{(\beta_{f'}\#_0 Gf) \\ \#_1(Hf'\#_0\beta_f)}"] \\
Fz \arrow[r, "\alpha_z"'] & Gz \arrow[r, "\beta_z"'] & Hz
\end{tikzcd}
$$

$$
\Rrightarrow \substack{(\beta_z\#_0\alpha^2_{f',f}) \\ \#_1(((\beta_{f'}\#_0 Gf) \\ \#_1(Hf'\#_0\beta_f))\#_0\alpha_x)}
$$

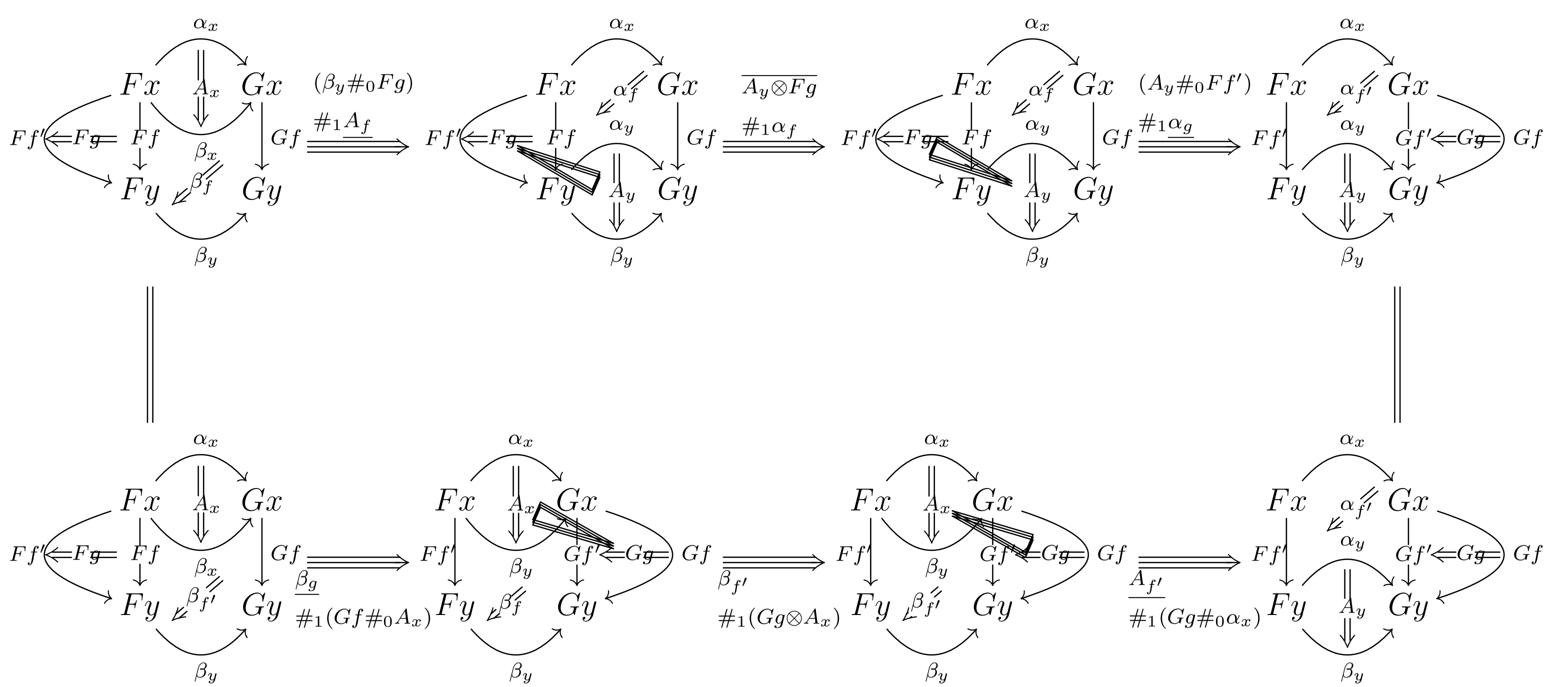
$=$

$$
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "F(f'\#_0 f)"'] & Gx \arrow[r, "\beta_x"] \arrow[d, bend right=30, "G(f'\#_0 f)"'] \arrow[d, bend left=30, "Gf'\#_0 Gf"] \arrow[d, phantom, "\Swarrow G^2_{f',f}"] \arrow[dl, phantom, "\Swarrow \alpha_{f'\#_0 f}"] & Hx \arrow[d, "Hf'\#_0 Hf"] \arrow[dl, phantom, "\Swarrow \substack{(\beta_{f'}\#_0 Gf) \\ \#_1(Hf'\#_0\beta_f)}"] \\
Fz \arrow[r, "\alpha_z"'] & Gz \arrow[r, "\beta_z"'] & Hz
\end{tikzcd}
\xRrightarrow[\substack{(\beta_z\#_0\alpha_{f'\#_0 f}) \\ \#_1(\beta^2_{f',f}\#_0\alpha_x)}]{}
\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "F(f'\#_0 f)"'] & Gx \arrow[r, "\beta_x"] \arrow[d, "G(f'\#_0 f)"] \arrow[dl, phantom, "\Swarrow \alpha_{f'\#_0 f}"] & Hx \arrow[d, "H(f'\#_0 f)"] \arrow[d, bend left=60, "Hf'\#_0 Hf"] \arrow[d, phantom, bend left=30, "\Leftarrow H^2_{f',f}"] \arrow[dl, phantom, "\Swarrow \beta_{f'\#_0 f}"] \\
Fz \arrow[r, "\alpha_z"'] & Gz \arrow[r, "\beta_z"'] & Hz
\end{tikzcd}
\qquad (279)
$$



(283)

Compatibility of the modification A with the cocycles of F, G, α, β



(284)

Compatibility of 2-cells with A, α and β

C.10. Remark. *According to C.9 a perturbation is given by a 3-cell in $\mathbb{H}$*

$$\begin{tikzcd} Fx \arrow[r, bend left, "\alpha_x"] \arrow[r, bend right, "\beta_x"'] \arrow[r, phantom, "\Downarrow A_x"] & Gx \arrow[r, Rrightarrow, "\sigma_x"] & Fx \arrow[r, bend left, "\alpha_x"] \arrow[r, bend right, "\beta_x"'] \arrow[r, phantom, "\Downarrow B_x"] & Gx \end{tikzcd} \tag{285}$$

for each 0-cell x in $\mathbb{G}$ such that

$$\begin{tikzcd}[column sep=large] Fx \arrow[r, bend left, "\alpha_x"] \arrow[r, bend right, "\beta_x"'] \arrow[r, phantom, "\Downarrow A_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\swarrow \beta_f"] \arrow[rr, phantom, "\overset{\beta_f \#_1(Gf\#_0\sigma_x)}{\Rrightarrow}"] & & Fx \arrow[r, bend left, "\alpha_x"] \arrow[r, bend right, "\beta_x"'] \arrow[r, phantom, "\Downarrow B_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\swarrow \beta_f"] \\ Fy \arrow[r, bend right, "\beta_y"'] \arrow[d, phantom, "\Downarrow A_f"] & Gy & & Fy \arrow[r, bend right, "\beta_y"'] \arrow[d, phantom, "\Downarrow B_f"] & Gy \\ Fx \arrow[r, bend left, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\swarrow \alpha_f"] \arrow[rr, phantom, "\underset{(\sigma_y\#_0Ff)\#_1\alpha_f}{\Rrightarrow}"] & & Fx \arrow[r, bend left, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \arrow[dl, phantom, "\swarrow \alpha_f"] \\ Fy \arrow[r, bend left, "\alpha_y"] \arrow[r, bend right, "\beta_y"'] \arrow[r, phantom, "\Downarrow A_y"] & Gy & & Fy \arrow[r, bend left, "\alpha_y"] \arrow[r, bend right, "\beta_y"'] \arrow[r, phantom, "\Downarrow B_y"] & Gy \end{tikzcd} \tag{286}$$

commutes.

C.11. Lemma. *A perturbation $\sigma\colon A \longrightarrow B$ fulfilling the conditions of C.8 is a 3-transfor in the sense of [Crans 1999].* □

D. More on Fibrations

D.1. Lemma. *Given fibrations of categories $p'\colon \mathsf{E}' \longrightarrow \mathsf{E}$ and $p\colon \mathsf{E} \longrightarrow \mathsf{B}$ their composite $pp'\colon \mathsf{E}' \longrightarrow \mathsf{B}$ is also a fibration.*

Proof Let X be an object in E' and $a\colon Y \longrightarrow pp'X$ an arrow in B, then there is a Cartesian lift $\overline{a}\colon a^*(qX) \longrightarrow qX$ of a along p; this arrow can be further lifted along p' to $\overline{\overline{a}}\colon \overline{a^*}a^*(X) \longrightarrow X$. Now, assume there is an $f\colon A \longrightarrow X$ in E' and a v with $pp'(f) = av$. There is unique $\langle v\rangle$ over v with $\overline{a}\langle v\rangle = q(f)$; and there is a unique $\langle\langle v\rangle\rangle$ over $\langle v\rangle$ with $\overline{\overline{a}}\langle\langle v\rangle\rangle = f$. Hence a has a Cartesian lift along pp'. □

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